Math Around Us

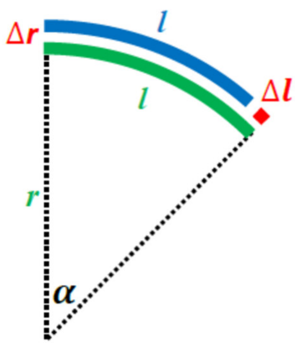
Senior League. 2022/11/03

Solutions

Problem 1

For the upcoming examination, the teacher printed out 40 tickets, assembled the sheets in a straight stack, and cut them to the desired size from all the four sides (see the top figure). Some of the newly cut edges of the sheets stuck together, which could slow down handing them out. So the teacher wants to separate them by shifting the stacked sheets in such a way that each ticket is displaced relative to the underlying one by the same distance, say 1 mm, and in the same direction (see the bottom figure). Suggest a method to do so *quickly* (say, in 5 s) and *only with her bare hands* (without using any objects), but with the help of geometry. Explain why all displacements of the tickets will be the same.

Answer: Grip and squeeze the stack on the left with your left hand. Bend it into an arc and fix the bent part of the stack with your right hand. Then, without loosening your right-hand grip, straighten the stack on the left with your left hand. You'll get the desired "staircase" of sheets. To decrease its slope and increase the relative shift of the sheets, repeat the operation as many times as needed.



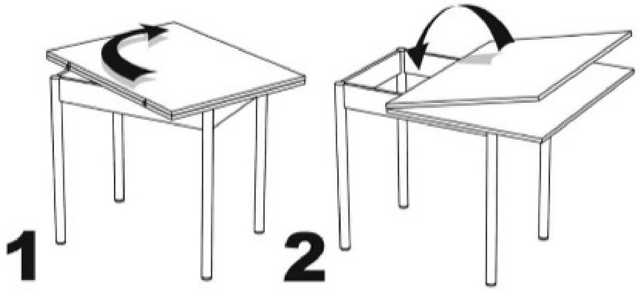
Explanation. Consider two adjacent sheets of width l and thickness Δr each. Bend them into circular arcs keeping their left edges fixed. The radii of the arcs will be r (in the figure, the inner, green arc) and $r+\Delta r$ (the outer, blue arc). If the angular measure of the inner sheet is α rad, then we have

$$l = \alpha r,\ \ l + \Delta l = \alpha(r + \Delta r) \Longrightarrow \Delta l = \alpha\Delta r,$$

which means that the right edge of the upper sheet will be shifted by Δl from the edge of the lower sheet to the left (horizontal after straightening the stack when fixing its right edge), and so the displacement of any two adjacent sheets in the stack will be the same.

Problem 2

The tabletop of a foldable dining table consists of two rectangular halves joined along a side 60 cm long by a hinge. All the four edges of the folded tabletop jut out over the rectangular frame of the table (its lower supporting part with legs) by the same distance. To unfold the table, as shown in the figure, we must:



1) rotate the tabletop in the horizontal plane;

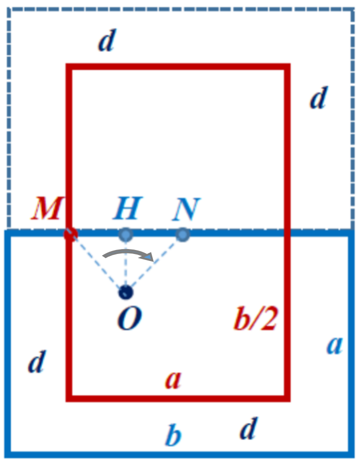
2) open it like a book.

A. Find the position of the pivot around which the folded tabletop is rotated at the first step. As the answer, give the distance from the pivot to the **nearest** edge of the folded tabletop.

B. One side of the unfolded tabletop is 60 cm. What is the length of its other side if all its four edges jut out over the frame by the same distance again, as they did before the table was opened?

Answer: **A**. 10 cm; **B**. 80 cm.

Solution. Let the lengths of the sides of the folded table be a and b=60.



A. Obviously, the angle of rotation is 90°. Also, since the rotation takes the edge of the folded table with the hinge on it into the midline of the unfolded table, it takes the midpoint M of this edge into the midpoint N of the midline, i.e., into the center of the tabletop (it's the same point in both positions). It follows that triangle OMN, where O is the pivot point (the center of rotation), is a right isosceles triangle with $OM = ON$ and $\angle MON = 90°$ (see the figure).

Its hypotenuse $MN = a/2$ and the distance OH from O to MN is $a/4$, so the distance to the opposite edge of the folded tabletop is $3a/4$ and to the other two edges, $\frac{b}{2} \pm \frac{a}{4}$. Below, from the answer to question B, we'll see that $a = 40$. Therefore, the four distances from O to the sides are 10, 30, 40, and 20. The shortest of them is $OH = 10$ (cm).

B. The side lengths of the unfolded table are b and $2a$. Since the edges of the unfolded table are the same distance d apart from the corresponding edges of the folded table, we have:

$b - a = 2d = 2a - b \Longrightarrow 2a = 4b/3 = 80.$

Problem 3

It is known that if two figures are similar with ratio k, then the ratio of their corresponding linear dimensions is k, the ratio of areas is k^2, and of volumes, k^3.

I. A conical glass is filled to the brim with lemonade (see the top figure). After Humpty drank some lemonade from the glass, the height of the lemonade level in its conical part decreased by ¼. Then Dumpty drank the rest.

A. Which of them drank more lemonade and by how many times?

B. By what fraction of the lemonade height should have Humpty decreased its level in the glass in order to split the drink fairly, in half?

II. Divisions on the conical part of a flask (see the bottom figure) are drawn so that after filling the flask, the volume of the liquid between the neighboring divisions is the same. The distance between the first (lower) and the second divisions on a given flask is 10 mm, and between the second and third ones, 13 mm.

C. Find the distance between the third and fourth divisions on the flask.

D. What is the greatest possible number of divisions for the given flask? For example, on the flask in the figure, the number of divisions is four.

Answers:

A. Humpty, $37/27 = 1{,}37\ldots$; **B.** $1 - \frac{1}{\sqrt[3]{2}} = 0{,}20\ldots$; **C.** $\approx 20{,}4$ mm; **D.** 4.

Solution. **I.** We can assume that the height of the conical part of the glass is 1.

A. The first volume is to the second as $(1^3 - (3/4)^3) : (3/4)^3 = 37 : 27$.

B. If x is the unknown fraction, then $2(1-x)^3 = 1 \Longrightarrow x = 1 - \frac{1}{\sqrt[3]{2}} = 0{,}206\ldots$

II. Let us draw horizontal planes through all the divisions on the flask and imagine it is extended to a complete cone whose base is on the plane through the first division. Let the height of this cone be 1. Denote by x the distance from its base to the second plane and by $a_i x$ the distance from the base to the $(i+1)$th plane. Then the volume of the cone's part between the base and the $(i+1)$th plane is i times the volume between the base and the 2nd plane. Therefore,

$$i(1^3 - (1-x)^3) = 1^3 - (1 - a_i x)^3 \Longrightarrow a_i^3 x^2 - 3\left(a_i^2 - i\right)x + 3(a_i - i) = 0.$$

For $i = 2$ we have $a_2 \equiv 1 + 1{,}3 = 2{,}3$, so for $x < 0{,}5$ we get the equation

$$10{,}167x^2 - 9{,}87x + 0{,}9 = 0 \Longrightarrow x = \frac{9{,}87 - 7{,}7985\ldots}{20{,}334} = 0{,}10187\ldots \approx 0{,}10.$$

C. If $i = 3$, then, by setting $x = 0{,}10187\ldots$, we obtain

$$a_3 x = 1 + \sqrt[3]{3(1-(1-x)^3) - 1} \Longrightarrow$$

$$a_3 = 0{,}10187\ldots^{-1}\left(1 + \sqrt[3]{3(1-(1-0{,}10187\ldots)^3) - 1}\right) = 4{,}3424\ldots 4{,}34.$$

It follows that the distance in question is $10(a_3 - a_2) \approx 20{,}4$ *мм*.

D. Let's find the largest integer i such that $a_i x < 1$ for the particular given flask:

$$1 + \sqrt[3]{i(1-(1-x)^3) - 1} < 1 \Leftrightarrow i(1-(1-0{,}10187\ldots)^3) < 1 \Leftrightarrow i < 3{,}6\ldots$$

Therefore, the desired i equals 3, and the greatest possible number of divisions equals $i + 1 = 4$.