Math Around Us

Senior League 2021/11/04

Problem 1

The figure shows a turnstile for the passage of people strictly one at a time. It consists of three bars fixed on a swivel drum attached to the inclined upper part of the supporting stand. As the drum spins, the bars cyclically change positions, each taking the vertical position at a certain moment. When the turnstile is in the standby mode, one of its bars is horizontal and the other two are directed downward, but lie in a plane that for some reason is not vertical (the figure clearly shows that this plane is not parallel to the post).

A. Is there a geometric reason why the specified plane is not vertical?

B. What is the angle of the upper (inclined) part of the rack with the vertical?

C. Find the angle that the planks make with each other.

D. What is the angle of the plane described in the condition with the vertical?

Answer: A. Yes. **B.** 45°. **C.** 75,5 … °. **D.** 18,4 … °.

Solution. The turnstile strips describe a conical surface in which the generators rotate around an axis perpendicular to the inclined plane of the drum.

A. The plane passing through the axis of rotation and the turnstile post contains exactly two generatrices (perpendicular to each other):

- horizontal (coinciding with the upper bar in the waiting state);
- vertical (coinciding with the bar at the time of its parallelism to the rack).

The projections onto this plane of all other generators, including the two coinciding projections of the lower bars in the waiting state, turn out to be non-vertical.

B. The axis of rotation makes an angle of $90°/2 = 45°$ with the slats (forming a cone) —the same angle is made by the stand with its inclined plane.

C. The projections of the strips onto a plane parallel to the plane of the drum, say, of length 1, form an angle of $360° : 3 = 120°$ with each other. Therefore, the isosceles triangle formed by the planks has sides $\sqrt{2}, \sqrt{2}, \sqrt{3}$ and an apex angle

$$2\arcsin\frac{\sqrt{3}/2}{2} = 2\arcsin\frac{\sqrt{6}}{4} = 75{,}5\ldots° = \arccos\frac{1}{4} = \arccos\frac{\sqrt{2}^2 + \sqrt{2}^2 - \sqrt{3}^2}{2\cdot\sqrt{2}\cdot\sqrt{2}}.$$

D. From the previous construction, we find that the desired angle is equal to

$$45° - \operatorname{arctg}\frac{1}{2} = 45° - 26{,}5\ldots° = 18{,}4\ldots°.$$

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Problem 2

A group of 5 tourists are going to get from one point to another along the highway using their 3 bicycles. All of them plan to start at 6:00. Any of those who ride a bicycle, by agreement with the others, can leave it in a certain place and continue the trip by foot. Any tourist, who travels by foot, having reached a bicycle left by another tourist, can take it and proceed by riding. Such switches from walking to riding a bike and vice versa can be done many times. The walking speed and riding speed of each tourist are constant. It takes 5 hours for each tourist to travel the entire distance by foot and 2 hours by bicycle. At what earliest time can the last traveler arrive at the destination?

Answer: 9:12.

Solution. Let the tourists be numbered 1-5.

1. The sum of the times t_1, t_2, t_3, t_4, t_5 of the movement of all 5 tourists is

$$2 + 2 + 2 + 5 + 5 = 16\ (h),$$

therefore, for the largest of these times, we have the lower bound

$$\max\{t_1, t_2, t_3, t_4, t_5\} \geq \frac{t_1 + t_2 + t_3 + t_4 + t_5}{5} = \frac{16}{5} = 3\frac{1}{5}(h).$$

2. Equality in this estimate is achieved if all travel times of tourists are the same. This situation can be realized by dividing the entire path into five equal parts and organizing the movement of tourists as follows:

- everyone starts moving at exactly 6:00;
- each part of the path on bicycles is ridden by 3 tourists (the rest are on foot) with certain numbers, namely:
  - the first part — 1st, 2nd and 3rd;
  - the second part — the 2nd, 3rd and 4th (the 4th tourist takes the 1st bike);
  - the third part — the 3rd, 4th and 5th (the 5th tourist takes the 2nd bike);
  - the fourth part — the 4th, 5th and 1st (the 1st tourist takes the 3rd bike);
  - the fifth part — the 5th, 1st and 2nd (the 2nd tourist takes the 4th bike);
- everyone gets to their destination at exactly 9:12.

Note that in this situation, the moment of taking each bike by the next tourist comes later than the moment it was left by the previous tourist.

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Problem 3

A hundred senators plan to pass a law consisting of seven items at the Senate meeting. For each item of the law, at least two alternative wordings have been proposed. It is possible that some of these wordings contradict to each other. According to the regulations, a certain wording of an item passes if it gets at least k of 100 votes, where k is a fixed positive integer, the same for all items. Before the meeting, each senator selects one wording for each of the seven items in such a way that no two of them contradict each other, and at the Senate meeting all senators vote for the options they have selected in advance. What smallest value of k guarantees that if the law as a whole, in all its seven items, passes, then no two of its items will contradict each other?

Answer: 86.

Solution. Let's consider two cases.

1. First, let $k = 86$. Then if each clause of the law is adopted by at least k votes, then no more than 14 senators can vote against each clause, and in total - no more than $14 \cdot 7 = 98$. Therefore, there will certainly be $100 - 98 \geq 1$ senator who voted for all points of the adopted law at once. This means that the set of 7 accepted statements will turn out to be consistent.

2. Now let $k < 86$. Then even k votes for each of the 7 points may not be enough for the consistency of the adopted law as a whole. Indeed, let the law aims at ordering 7 values $a_1, a_2, \ldots, a_7$, and the first versions of the wording of all 7 points form a contradictory (cyclic) list of statements

$$a_1 < a_2, \quad a_2 < a_3, \quad \ldots, \quad a_6 < a_7, \quad a_7 < a_1,$$

and the second options are the same, but with opposite inequalities. Let all 100 senators split into 7 factions: in factions with numbers 1-6, 15 people each, and in the 7th faction there are 10 people. Suppose that for each $i = 1, \ldots, 6$ senators of the i-th faction voted for the first versions of the wording of all items, except for item i, in which they supported the second version of the wording. Then, for each senator, the set of his formulations, containing 6 formulations of the first type and one of the second, is consistent. However, the entire adopted law as a whole, each clause of which was approved by at least 85 votes, is contradictory, since it consists of only the first type of wording.