MATS Olympiad 2021. Senior league. Problem 1.

A beginner football-player Vassily can run with a ball with the velocity of v_1=5 m/s and hit the ball at rest, so the ball flies with the initial velocity of v_2=10 m/s by any angle to the horizon. The boy can stop himself and the ball instantly. What is the minimal time required to move the ball from one sideline to the other? The ball stops immediately after landing after the flight and it is not allowed to touch it after that. The width of the field is 64 m, gravity acceleration is 10 m/s^2, there is no air resistance.

Solution

It is clear that the player should go right across the field to have distance to cover L=64 m, not more. As the velocity of the ball in flight is larger than the velocity of running, it is reasonable to fit the ball at some point to let it fly and land at the side line.

Consider a time of the crossing as the function of the angle α between the ball's initial velocity and the horizon:

$$T = \frac{L - v_2^2 \sin 2\alpha / g}{v_1} + \frac{2 v_2 \sin\alpha}{g} \qquad (1)$$

To find the minimum of this function at the segment $\alpha \in [0; \pi/2]$ we calculate the derivative

$$\frac{dT}{d\alpha} = \frac{v_2}{g}\left[-2\frac{v_2}{v_1}\cos 2\alpha + 2\cos\alpha\right]$$

At check where it equals zero. We obtain the equation

$$n \cos 2\alpha = \cos\alpha, n = \frac{v_2}{v_1}$$

One of its roots

$$\alpha = \arccos\frac{1+\sqrt{1+8n^2}}{4n}$$

for n>1 belongs to $[0, \pi/4]$ and corresponds to a minimum and had physical meaning. Substituting this solution to (1) gives the minimal time, for given values $T_{min} = 12.06c$

MATS Olympiad 2021. Senior league. Problem 2.

A boy Vassily found a funnel that has an exit hole diameter of 2 cm. He had to fill a two-liter bottle by use of this funnel, the bottle neck diameter is 1 cm. The boy managed to fix the funnel, so its exit hole is above the bottle neck by 45cm. Vassily found out that with some care no water goes past the bottle. What is the minimal time required for filling the two-liter bottle with the described setup, if no water goes past the bottle?

Solution

Let U_0 be the water velocity at the exit of the funnel. Between the funnel and the bottle, we observe the free fall of the jet, so the energy conservation law gives the velocity at the enter of the bottle $U = \sqrt{U_0^2 + 2gh}$. Since the volume of the water crossing any cross-section per unit time is the same, we have the equality (continuity equation):

$$U_0 \frac{\pi D^2}{4} = U \frac{\pi d_1^2}{4} \Leftrightarrow U_0 D^2 = \sqrt{U_0^2 + 2gh}\, d_1^2,$$

$$U_0 D^2 = U d_1^2 \Leftrightarrow U = \frac{U_0 D^2}{d_1^2} \Leftrightarrow U_0 = \frac{d_1^2}{D^2}\sqrt{U_0 + 2gh} \Leftrightarrow \frac{U_0^2 D^4}{d_1^4} = U_0^2 + 2gh \Leftrightarrow$$

$$\frac{U_0^2(D^4 - d_1^4)}{d_1^4} = 2gh \Leftrightarrow U_0 = d_1^2 * \frac{\sqrt{2gh}}{\sqrt{D^4 - d_1^4}}$$

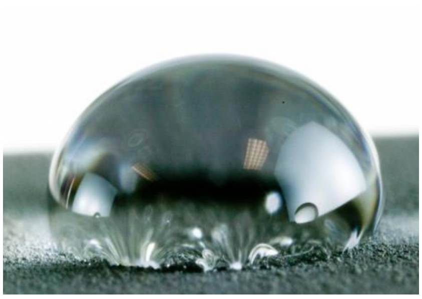
where D is the diameter of the exit of the funnel, d_1 is the jet diameter at the bottle enter. To collect all water, we demand $d_1 \leq d,$ where d is the diameter of the bottle enter. Thus,

$$U_0 \leq \frac{\sqrt{2gh}d^2}{\sqrt{D^4 - d^4}},$$

and the time required for the collection of the volume V is

$$T = \frac{V}{U_0 \pi D^2/4} \geq \frac{4V\sqrt{D^4 - d^4}}{\pi D^2 d^2 \sqrt{2gh}} = 8{,}22\ s$$

MATS Olympiad 2021. Senior league. Problem 3.



A laboratory deals with experiments on evaporation. First, they placed a large enough droplet on a plate and found out that the droplet remained semispherical throughout the experiment and its radius depended on time as r(t)=R/(1+Dt), where R is the initial droplet radius, D is a constant. In the second experiment, they put an initial droplet on the plate and added droplets of the volume of V by equal time intervals T. The liquid become a semispherical droplet immediately after each adding. Answer the following questions about the second experiment.

a. Let the initial volume is such that the dependence of the droplet radius on time is periodic. What is the maximal and minimal liquid volume during the experiment in this case? Make your calculations for V=1 mm^3, D=1 s^{-1}, T=1 s.
b. Let V, D, T be the same as for subproblem a, and the volume of the initial droplet be 8 mm^3. What is the liquid volume after adding the 4th droplet?
c. Prove that for any initial droplet volume the dependence of radius on time tends to the periodic one, described in subproblem a.

Assume that the evaporation is always the same as in the first experiment. The volume of a hemisphere of the radius of r is $2pr^3/3$.

Solution

a. For the periodic regime. The volume decrease during the evaporation is compensated by the volume of the adding droplet. Let R_0 be the initial radius, then:

$2pR_0^3/3 - 2pR_0^3/(3(1+DT)^3)=V$,

so

$$R_0 = \left\{\frac{3V}{2\pi}\left[\frac{(1+DT)^3}{(1+DT)^3-1}\right]\right\}^{1/3}$$

Maximal volume is the initial one, it occurs right after adding the droplet. It corresponds to radius found above

$$V_+ = V\frac{(1+DT)^3}{(1+DT)^3-1} = \frac{8}{7}mm^3 = 1.143mm^3$$

Minimal volume is less by V and equals

$$V_- = V\frac{1}{(1+DT)^3-1} = \frac{1}{7}mm^3 = 0.143mm^3$$

b. The values V_n at instants nT (right after adding of the n-th droplets) form the sequence given by the following relation:

$$V_{n+1} = \frac{1}{(1+DT)^3} V_n + V$$

The required value can be found by direct calculation or by the formula of the common member:

$$V_n = Cq^n + \frac{V}{1-q}, \qquad q = \frac{1}{(1+DT)^3}, \qquad |q| < 1, \qquad \mathrm{C} = V_0 - \frac{V}{1-q}$$

For given values q=1/8, and C=48/7 mm^3, so

$$V_4 = \frac{6}{8^3 \cdot 7} + \frac{8}{7} = \frac{4102}{3584} = 1.144 mm^3$$

c. The amount of the evaporated liquid *v* monotonically depends on the initial radius:

$$v = \frac{2\pi R^3}{3} \frac{(1+DT)^3 - 1}{(1+DT)^3}$$

If v<V, then the liquid volume for the next droplet would be larger than for the given one, and the evaporated volume would be larger than for the previous step.

For v>V, we have the opposite situation. During the cycle the volume would decrease and the evaporated volume as well.

The sequence V_n at instants nT (right after adding of the n-th droplet) forms the sequence given by the expression:

$$V_{n+1} = \frac{1}{(1+DT)^3} V_n + V$$

The formula for the h-th member is

$$V_n = Cq^n + \frac{V}{1-q}, \qquad q = \frac{1}{(1+DT)^3}, \qquad |q| < 1, \qquad \mathrm{C} = V_0 - \frac{V}{1-q}$$

where V_0 is the initial volume. While *n* increases the first term vanishes and V_n tends to $V/(1-q)=V_+$, so the periodic regime occurs.