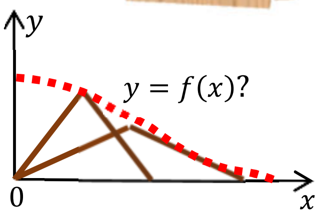
Math Around Us

Senior League 2020/11/23

Problem 3

A folding book-door consists of two movable halves joined by a hinge (see the top figure). One half rotates around a fixed axis, like an ordinary door, and the top end of the opposite edge of the other half has a roller that slides along a fixed runner. The bottom figure shows the top view of this door: its two halves look like two segments of length 1 each, the door axis is at the origin, and the runner goes along the positive x half-axis. The region swept by this pair of segments on the (x, y) plane as their rightmost endpoint slides along the x axis is bounded by the coordinate axes and a curve.



Let us *assume* that this curve consists of two arcs, the left one and the right one, which are specified, respectively, by the equations

$x^\alpha + y^\alpha = \beta,\ \ 0 \le x \le x_0,$ and $x^\gamma + y^\gamma = \delta,\ \ x_0 \le x \le 2,$

for certain positive values of $\alpha, \beta, \gamma, \delta$ and $x_0 \in (0; 2)$.

A. Find the values of α and β.

B. Find the value of x_0, at which the left arc is joined to the right arc.

C. Find the values of γ and δ.

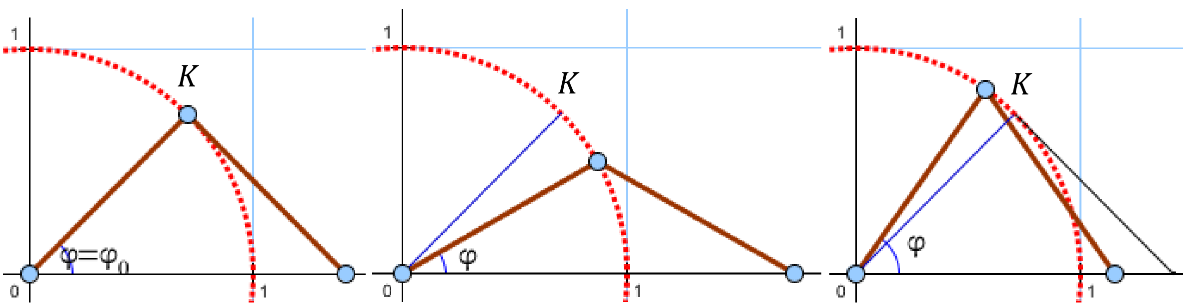
D. Is our assumption about the curve correct?

ANSWER: **A.** $\alpha = 2,\ \beta = 1.$ **B.** $x_0 = 1/\sqrt{2}.$ **C.** $\gamma = 2/3,\ \delta = \sqrt[3]{4}.$ **D.** Yes, the assumption is correct.

SOLUTION. Denote by φ the angle between the x-axis and the left half of the door in our figure.

A. The left arc of the curve is formed by the hinge rotating around the origin, and so describes a quarter-circle with the equation $x^2 + y^2 = 1$, because for φ close to 90° the other half falls completely inside this circle and does not reach the curve. Therefore, $\alpha = 2,\ \beta = 1$.

B. When $\varphi = \varphi_0 \equiv 45°$, the right-hand half lies on the line tangent to the circle at the point $K\left(\frac{1}{\sqrt{2}}; \frac{1}{\sqrt{2}}\right)$, and so completely lies outside the circle except for the hinge, which is on the circle. The same is true for $\varphi < \varphi_0$. As for $\varphi > \varphi_0$, it crosses the circle at a point below K, and so its piece outside the circle lies below the tangent at K. So the two arcs meet at K and $x_0 = 1/\sqrt{2}$.



C. Substituting the coordinates $\left(\frac{1}{\sqrt{2}};\frac{1}{\sqrt{2}}\right)$ and $(2;\ 0)$ of the endpoints of the right arc into its proposed equation, we obtain

$$1/\sqrt{2}^{\gamma}+1/\sqrt{2}^{\gamma}=\delta=2^{\gamma}\ \ \Rightarrow\ \ \gamma=2/3,\ \ \delta=\sqrt[3]{4}.$$

D. Let us prove that the curve defined by the equations in the statement with $\alpha=2,\beta=1,\gamma=2/3,\delta=\sqrt[3]{4}$ is the desired one, and so our assumption is true. Indeed, we have found its left half directly and its right half satisfies the following conditions:

1) Straight lines that extend the right half of the door form a family of lines given, for $\varphi_0\geq\varphi\geq 0$, by the equation

$$\frac{x}{\cos\varphi}+\frac{y}{\sin\varphi}=2.$$

2) Taking the derivative with respect to x of the equation that specifies the right arc of the curve in which y is regarded as a function of x, we obtain

$$\gamma x^{\gamma-1}+\gamma y^{\gamma-1}y'=0\ \Rightarrow y'=-(y/x)^{1-\gamma}=-(y/x)^{1/3}<0,\ x_0\leq x\leq 2,\ y_0\geq y\geq 0,$$

whence it follows that the function y decreases, and so its derivative y' increases. This means that the curve is convex (downward), that is, it lies above any of its tangents.

3) If a given point $(x;y)$ lies on the right arc of our curve, then the slope of the tangent at the given point equals $-k=-(y/x)^{1/3}$. Hence the line from the family with the same slope $-k=-\operatorname{tg}\varphi$ passes just through the given point, because

$$\frac{x}{\cos\varphi}+\frac{y}{\sin\varphi}=\frac{x}{1/\sqrt{1+k^2}}+\frac{y}{k/\sqrt{1+k^2}}=x^{\gamma}\sqrt{x^{\gamma}+y^{\gamma}}+y^{\gamma}\sqrt{x^{\gamma}+y^{\gamma}}=(2^{\gamma})^{3/2}=2.$$

So this line touches the curve at the given point.

4) Thus, the set of all tangents to this part of the curve coincides with the set of the corresponding positions of the door's right half; therefore, this half does not sweep any of the points above the curve, but sweeps all the points on the curve and below it.