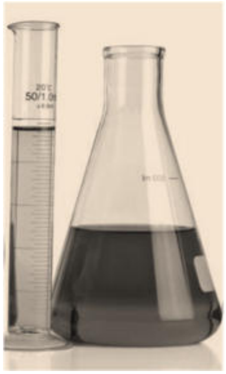
Math Around Us

Senior League 2020/11/23

Problem 2



After a chemistry lesson, the teacher emptied the flask with the acid she used. Now she wants to wash it with pure water from a beaker. She is going to divide the water in the beaker into n portions (not necessarily equal), then carefully rinse the flask with the first portion of water and empty it, then rinse it with the second portion of water and empty it again, and so on. Every time she empties the flask, the same amount of liquid remains on its walls.

A. For what number n of equal portions of water in the flask, 1, 2, 3, or 4, it will turn out the cleanest after washing in this manner (i.e. the leftover of acid in it will be the smallest)?

B. If the teacher divides the water into three portions, what should be the ratio of their volumes to ensure the cleanest flask in the end?

C. Is it true that the greater is the number n of equal portions, the cleaner is the flask in the end?

D. Is it true that as the number n of equal portions increases, the amount of acid that remains in the flask tends to zero?

ANSWER: **A.** 4. **B.** They must be equal. **B.** Yes. **Г.** No.

SOLUTION. Let's take for the unit volume the amount of liquid that remains on the flask's walls after it is emptied. Then, if we add pure water of volume of x into the flask, then the strength of the acid, that is, the amount of acid per a unit volume of solution will decrease $1+x$ times, and after we empty the flask again, the volume of acid in the remaining unit volume of solution will be $(1+x)^{-1}$.

A, C. (*The following nice solution was given by the Letovo School team.*) If the initial amount x of water is divided into $n(=3)$ equal portions, then in the end we will have $\left(1+\frac{x}{n}\right)^{-n}$ of acid. Let us prove that the amount of acid remaining after successive rinsing of the flask with $n+1$ equal portions of water, which is $\left(1+\frac{x}{n+1}\right)^{-(n+1)}$, is smaller. In other words, let us prove the inequality $\left(1+\frac{x}{n}\right)^{n} < \left(1+\frac{x}{n+1}\right)^{n+1}$.

On its left-hand side we have the product of n factors $1+\frac{x}{n}$. Imagine we add a factor equal to 1 to this product. The product will not change, but the sum of its factors will become equal to $n+1+x$, which is just the sum of the $n+1$ factors on the right-hand side of our inequality. But it is well known that the product of a given number of positive factors with a given sum is the greatest if the factors are equal (this is a version of AM-GM inequality), which completes the proof.

B. If the water is divided into n arbitrary portions, $x_1+\dots+x_n=x$, then by the remark above, we get the lower bound

$$\left((1+x_1)\dots(1+x_n)\right)^{-1} \geq \left(\frac{(1+x_1)+\dots+(1+x_n)}{n}\right)^{-n} = \left(1+\frac{x}{n}\right)^{-n}.$$

This inequality turns into equality just in the case of equal portions $x_1=\dots=x_n=x/n$.

D. If x is fixed and $n \to +\infty$, then, setting $t \equiv n/x \to +\infty$, we see that the smallest amount of acid that can be obtained is achieved for equal portions and is given by the limit

$$\left(1+\frac{x}{n}\right)^{-n} = \left(\left(1+\frac{1}{t}\right)^{t}\right)^{-x} \to e^{-x} > 0.$$