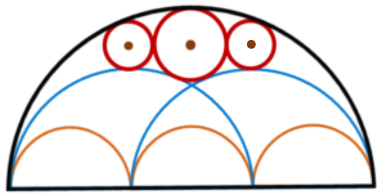
Math Around Us

Senior League 2020/11/23

Problem 1

In the photo you see a semicircular window of Odessa Philharmonic Hall decorated with circular arcs. At the very bottom of the window, there are three identical small semicircles, then two larger identical semicircles, and three circles inscribed in the upper part of the window bounded by its arc and the larger semicircles as shown in the figure under the photo.



A. Find the ratio of the radius of the largest inscribed circle to that of the window.

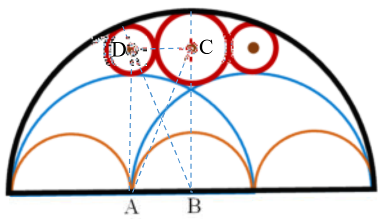
B. Is it true that the centers of three inscribed circles lie on a straight line?

C. What is the ratio of the radii of the largest and smallest inscribed circles?

ANSWER: **A.** 1 : 5. **B.** Yes, it is true. **C.** 3 : 2.

SOLUTION. Denote by $3a$ the radius of the largest semicircle (the window). Then the radii of the smaller and larger semicircles are a and $2a$, respectively. Let r be the radius of the largest of the three circles.

A. By the Pythagorean theorem for triangle ABC (see the figure), we have $a^2 + (3a - r)^2 = (2a + r)^2 \Rightarrow r = 3a/5$.



B (*The author of this question is Yuri Biletsky*). Let D be the fourth vertex of the rectangle $ABCD$, where $A, B,$ and C are the centers of the blue and orange semicircles and the big circle, and let $x, y,$ and z be the distances from D to the red circle, the blue semicircle, and the black semicircle that touch it (i.e. to their closest points) respectively. Then

$$a = AB = CD = x + r,$$
$$2a + y = AD = BC = 3a - r,$$
$$3a - z = BD = AC = 2a + r.$$

It follows that

$$x = y = z = a - r,$$

and so D turns out to be the center of the smaller red circle, whence the three centers in question colline.

C. The radius of the smaller circle equals

$$x = a - r = a - 3a/5 = 2a/5 = 2r/3.$$