Math Applied to Science Olympiad (MATS/Primate) Senior League.

Problem 1 In a thermodynamic process over 1 mol of a monoatomic ideal gas, its pressure p and volume V are connected by the relation

$$p = p_0 \left(1 - \frac{V}{V_0}\right),$$

for some constants p_0 and V_0. The gas slowly expands. Give justified answers to the following questions:

A. Does the gas temperature increase or decrease when volume $V = 9V_0/16$?

B. Does the gas receive or deliver heat at this moment?

Solution. The equation of state connects the gas temperature T with its pressure and volume (for 1 mol):

$$pV = RT.$$

It gives

$$T = \frac{pV}{R} = \frac{p_0 V}{R}\left(1 - \frac{V}{V_0}\right) = \frac{p_0 V_0}{R} x\,(1-x)\,, \quad x = \frac{V}{V_0}.$$

The temperature increases while $x < 1/2$, so for the given value $x = 9/16$ the temperature decreases.

To determine the direction of the heat transfer, consider the first of law of thermodynamics for a small volume change $\Delta V = V_0 \Delta x$. The amount of heat given to the gas is

$$\Delta Q = p\Delta V + c_v \Delta T,$$

where $c_v = 3R/2$ (the gas is monoatomic). The temperature change is (term of the order of Δx^2 are withdrawn)

$$\Delta T = \frac{p_0 V_0}{R}\left[\Delta x - \Delta(x^2)\right] = \frac{p_0 V_0}{R}\Delta x\,(1 - 2x)\,.$$

The first law of thermodynamics gives

$$\Delta Q = p_0 V_0 (1-x)\Delta x + \frac{c_v}{R} p_0 V_0 \Delta x (1-2x) = p_0 V_0 \Delta x \left[1 + \frac{c_v}{R} - x\left(1 + 2\frac{c_v}{R}\right)\right]$$

Since $\Delta x > 0$ (the gas expands), the term in brackets determines the sign of the heat transfer. The gas receives the heat ($\Delta Q > 0$), if

$$x < \frac{1 + c_v/R}{1 + 2c_v/R} = \frac{5}{8}.$$

For the given values of x, the condition holds, so the gas receives the heat.

Answer: The temperature decreases, the gas receives the heat.

Problem 2 The speed limit for vehicles in city M is 80 km/h, and all drivers strictly follow this rule. In addition, the city is famous for its traffic jams, so an average trip 30 km long takes 45 min. Give an upper estimation for the change of the average trip duration if the speed limit is lowered to 50 km/h while the trip length and the time lost in jams remain the same. Speeding is strictly forbidden. Give as precise estimation as you can and present the answer in percentage of the initial trip duration.

Solution Before the speed limit change, the average speed is 30/0.75=40 km/h. So the change of the limit will not affect the drivers who avoid the jams. It is clear that the major effect takes place for the drivers who partially stands still in a jam or goes at the speed limit.

The time spent in the jams is

$$\tau = T_1 - \frac{L}{V_1},$$

L is the trip length, V_1 and T_1 are maximal speed and the trip duration before the limit change.

After the change, the duration is (V_2 and T_2 are maximal speed and the trip duration before the limit change)

$$T_2 = \frac{L}{V_2} + \tau = T_1 + \frac{L}{V_2} - \frac{L}{V_1}$$

The relative change is

$$\varepsilon = \frac{T_2 - T_1}{T_1} = \frac{L(V_2 - V_1)}{T_1 V_1 V_2} = 30\%$$

Answer: 30%.

Problem 3 According to media reports, on July 3, 2020, the orbit of the International Space Station (ISS) was corrected off-schedule. As a result of the correction, the velocity of ISS changed by 0.5 m/s and its orbit radius increased by 900 m.

A. Using these data and given the Earth radius (6400 km) and the gravity acceleration at the Earth surface (9.8 m/s^2), estimate the height of ISS orbit over the Earth surface.

B. What accuracy (in meters) for the orbit radius change is required to ensure that the error in the orbit height determination is less than 100 km?

Solution Assume, the ISS orbit is circular before and after the correction. The gravity creates centripetal acceleration:

$$m\frac{v^2}{R} = G\frac{mM}{R^2},$$

where m, M are the station and Earth masses, G is gravitational constatnt, v is the velocsity of the station, and R is the orbit radius. For the orbit height d over the surface, we have

$$\frac{mv^2}{R_0 + d} = mg\frac{R_0^2}{(R_0 + d)^2}, \tag{1}$$

R_0 is Earth radius, $g = GM/R_0^2$ is gravity acceleration at the surface. Thus

$$v^2 = \frac{gR_0^2}{R_0 + d}.$$

Consider this relation for two situation 1) before the correction $d = H$, $v = v_0$, 2) after the correction $d = H + h$, $v = v_0 + \Delta v$:

$$v_0^2 = \frac{gR_0^2}{R_0 + H}, \quad (v_0 + \Delta v)^2 = \frac{gR_0^2}{R_0 + H + h} = \frac{gR_0^2}{R_0 + H}\left(1 + \frac{h}{R_0 + H}\right)^{-1}$$

Since $(1 + x)^\alpha \approx 1 + \alpha x$ for $x \ll 1$, and $|\Delta v| \ll v_0$, $h \ll (R_0 + H)$, we have

$$2v_0\Delta v = -\frac{gR_0^2}{(R_0 + H)^2}h \tag{2}$$

It is clear that $\Delta v < 0$. The problem statement gives the system of equations:

$$\frac{v_0(R_0+H)^2}{gR_0^2} = -\frac{1}{2}\frac{h}{\Delta v_0}$$
$$v_0^2 = \frac{gR_0^2}{R_0+H}.$$

Its silution is

$$H = R_0\left[\left(\frac{g}{4R_0}\right)^{1/3} \cdot \left(\frac{h^2}{(\Delta v)^2}\right)^{1/3} - 1\right] = 476 \text{ км}. \tag{3}$$

We withdrawn the following terms $(\Delta v/v_0)^2 \sim 10^{-8}$ и $(h/R_0)^2 \sim 10^{-7}$ during derivation of the equation (2). This approximation are negligible and it can be used for accuracy estimation. Taking linear approximation of (3) by differentiation (assume all values but h are known presicely and fixed):

$$\frac{dH}{dh} = \frac{2}{3}\left(\frac{g}{4(\Delta v)^2}\right)^{1/3} R_0^{2/3} h^{-1/3} \approx 5100.$$

Change of h by Δh leads to change of H by ΔH:

$$\Delta H = \frac{dH}{dh}\Delta h \approx 5100\Delta h.$$

The accuracy of $\Delta H = 100$ km corresponds to $\Delta h = 10^5/5100 \approx 19.5$ m.

Remark The obtained answer is overestimated, the orbit height is between 320 and 420 km. The problem B clarifies low accuracy of the first answer: large error comes from rounding by media.

Answer: 476 km; 19.5 m.