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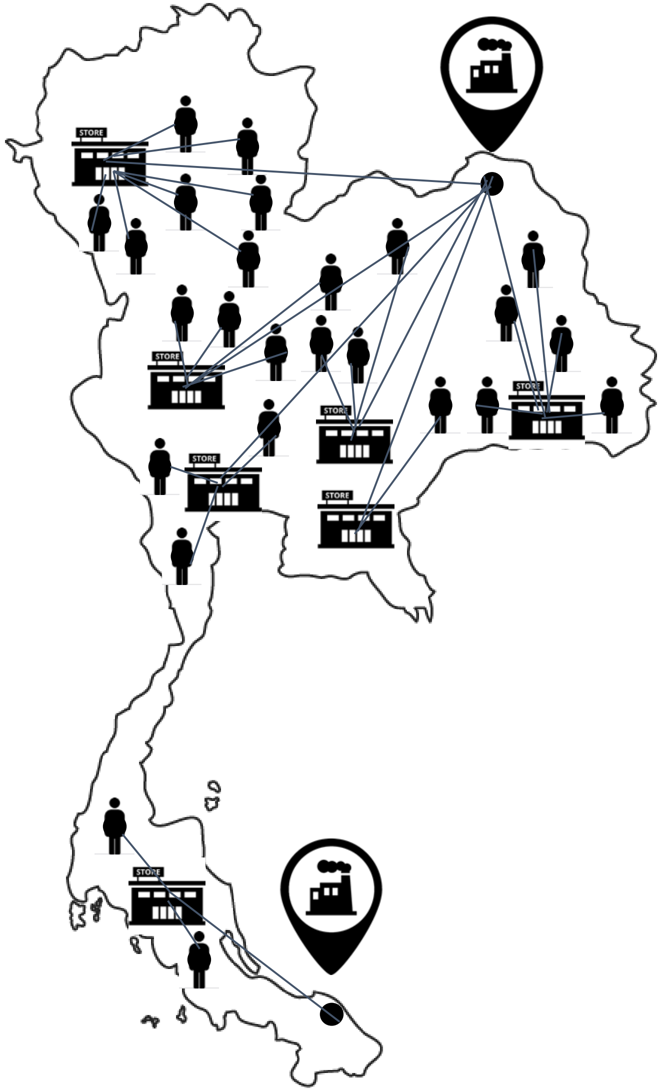
Improvement of Product Distribution Efficiency using Double-Weighted Flow Network on Lexicographic Product of Path Graph and Empty Graph

Introduction

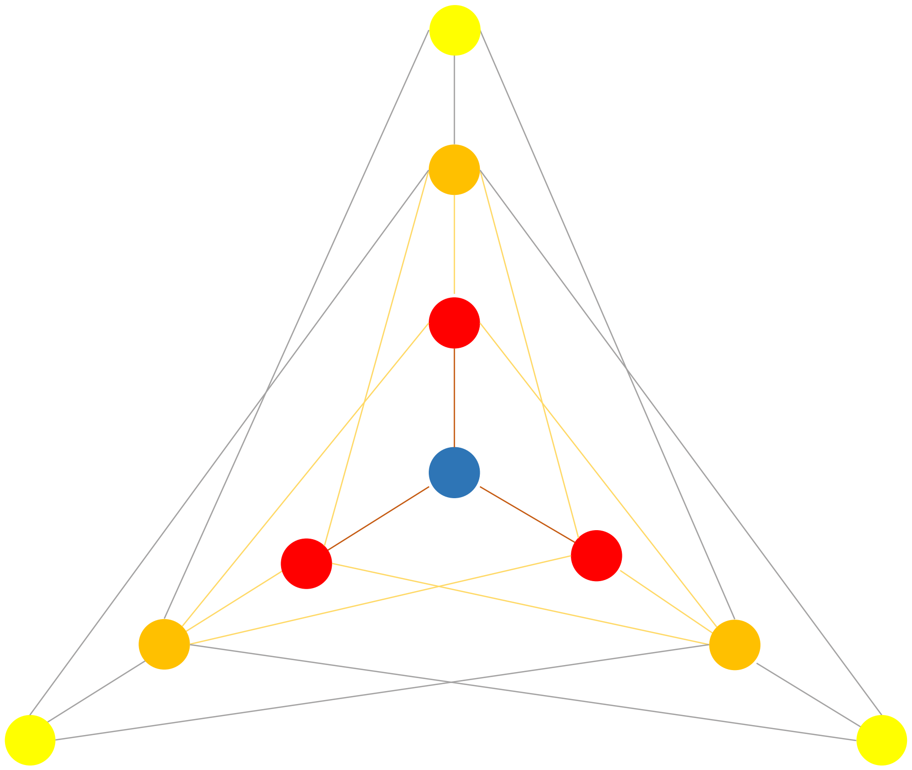


Most distribution systems start from the source of production, then the products will be sent to each authorized distribution center and to the consumers.

The product distribution system looks similar to a graph.

Introduction

The product distribution system begins to transport products to the biggest intermediary in the distribution channel and continues lowering until the consumers in which this specific behavior is similar to the flow network on Lexicographic product of graphs.



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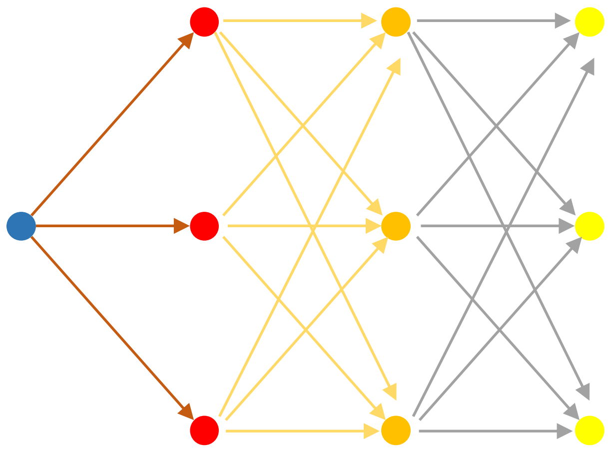


Lexicographic product of graphs

Lexicographic Product of Graphs

Path graph

Definition : The path graph P_n is a graph where n vertices can be listed in order u_1, u_2, ..., u_n such that the edges are $\{u_i, u_{i+1}\}$ where i = 1,2, ..., n.

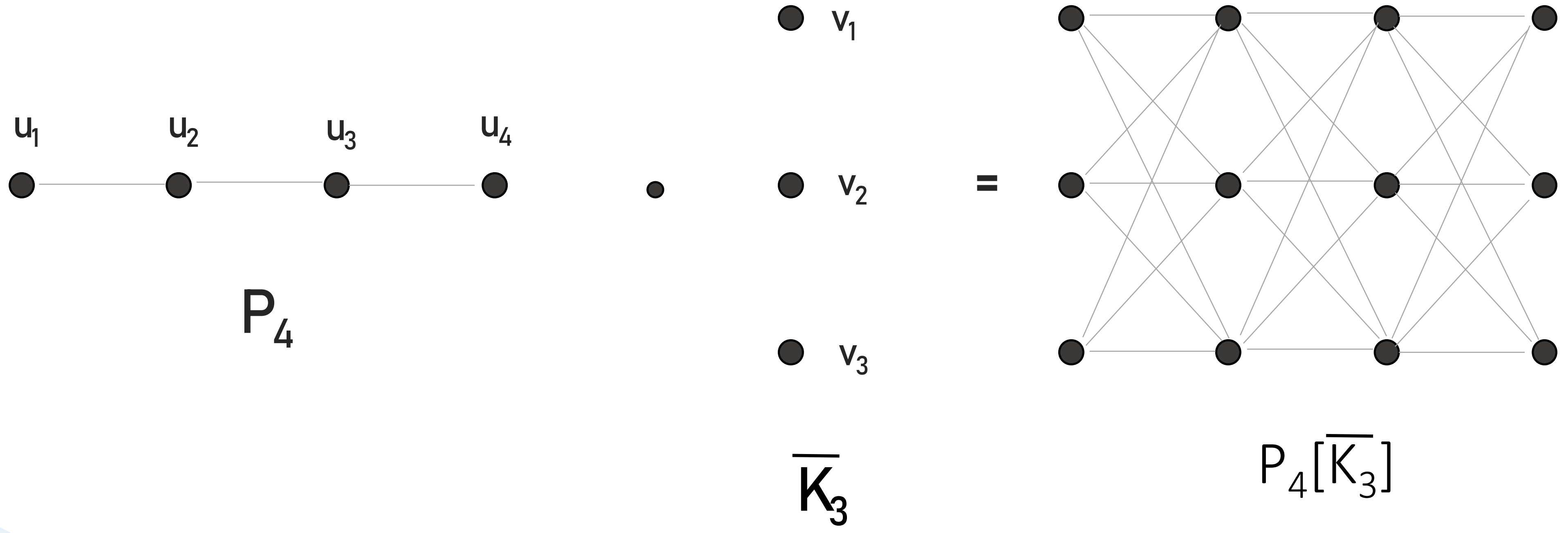
Example of P_4

Empty graph

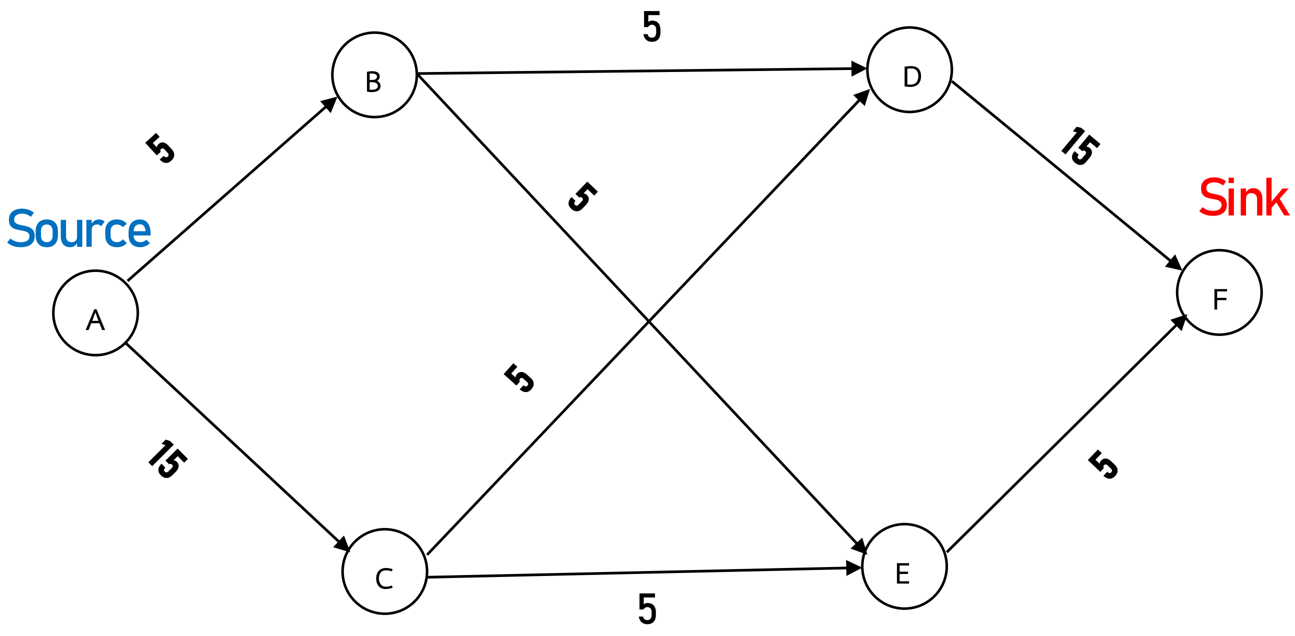
Definition : An empty graph $\overline{K_n}$ is a graph with n vertices and zero edges.

Example of $\overline{K_3}$

Lexicographic Product of Graphs



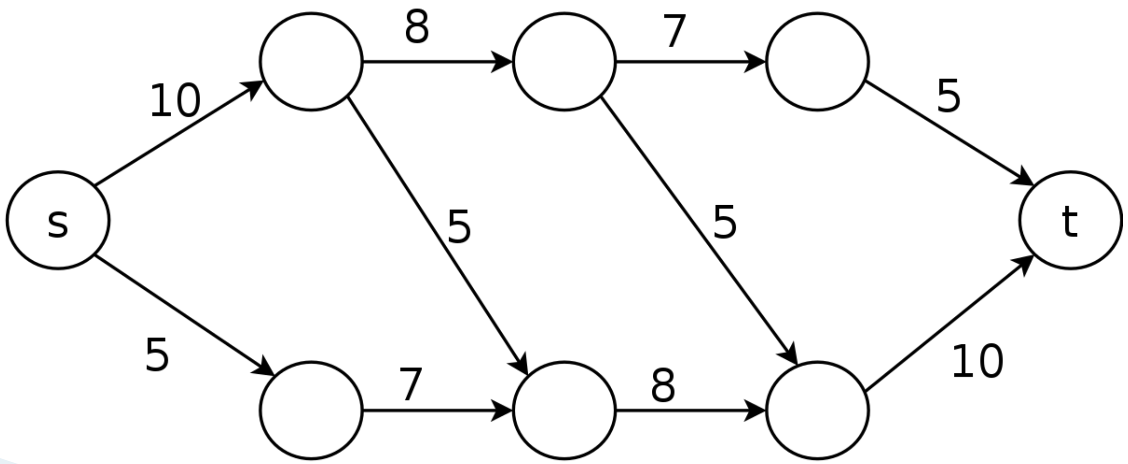
Flow network or transportation network



- Directed graph
- Capacity
- Source, which has only outgoing flow
- Sink, which has only incoming flow

Flow Network Definition

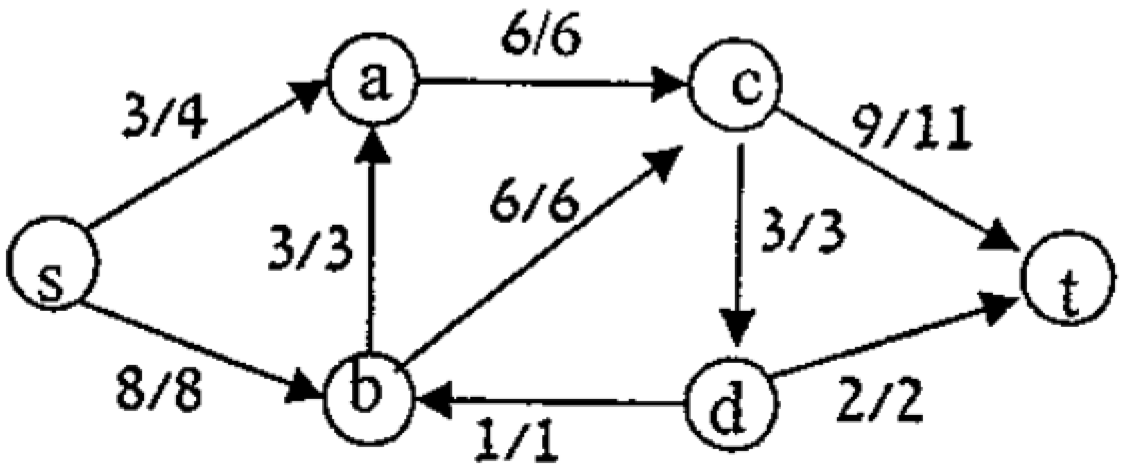
A flow network is a graph $G = (V, E)$, where V is a set of vertices and E is a set of V's edges – a subset of $V \times V$ – together with a non-negative function c: $V \times V \to \mathbb{R}_\infty$, called the capacity function. Without loss of generality, we may assume that if $(u, v) \in E$ then (v, u) is also a member of E, since if $(v, u) \notin E$ then we may add (v, u) to E and then set $c(v, u) = 0$.



Flow Definition

A flow is a function $f : V \times V \to \mathbb{R}$ that satisfies the following two constraints for all nodes u and v:

- *Skew symmetry*: Only encode the net flow of units between a pair of nodes u and v, that is: $f(u, v) = -f(v, u)$
- *Capacity constraint*: An arc's flow cannot exceed its capacity, that is: $f(u, v) \leq c(u, v)$.
- *Flow conservation*: The net flow *entering* the node v is 0, except for the source, which "produces" flow, and the sink, which "consumes" flow. That is: $x_f(v) = 0$ for all $v \in V \setminus \{s, t\}$



Definition of Efficiency

Definition 2.2.11 Denote $eff(f)$ is **efficiency** of flow f which

$$eff(f) = \frac{|f|}{\max|f|} + \frac{\min(T(f))}{T(f)}$$

and $\max(eff(f))$ is maximum efficiency in which some flow f in graph $B_{m,n}$ has.

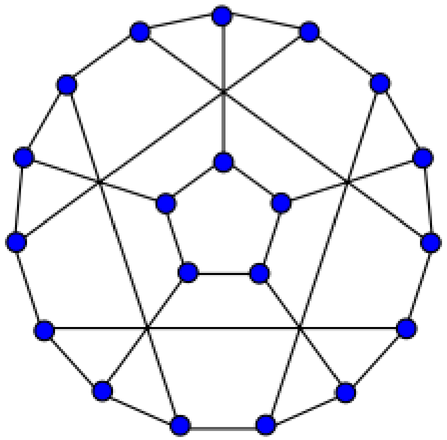
$$E_{dc} = E_o + E_w + E_p + E_t + E_{im} + E_u$$

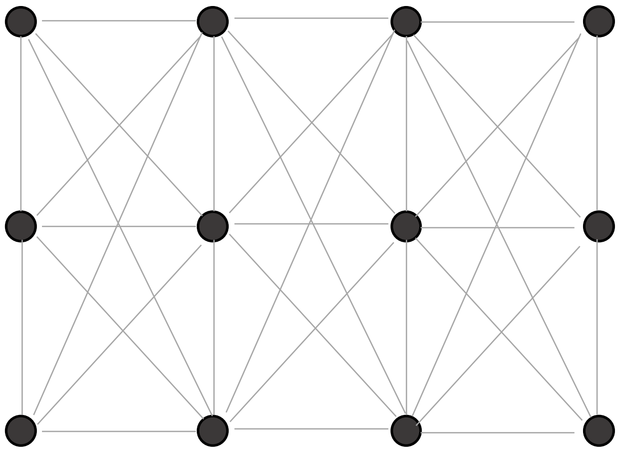
The efficiency of distribution channel

Milan Andrejic (2016)

Objective

- To deduce important theorems about how to find the most efficient way to transport supplies on $B_{m,n}$ which can be further applied to real life product distribution





Results

1. Minimum time flow solution in $B_{m,n}$

Theorem: Let $A_{i,j}$ be the minimum possible time from $v_{1,0}$ to $v_{i,j}$ then

$A_{i,j} = \min(A_{k,j-1} + t(k,i,j-1))$, $k = 1,2,3,\ldots,m$ for $j \geq 2$ when $A_{i,1} = t(1,i,0)$

and $\min(T(f)) = \min(A_{k,n} + t(k,1,n))$ for $k = 1,2,3,\ldots,m$

Proof: From Dijkstra Algorithm

Consider an arbitrary vertex $v_{x,j}$ only $(1,x,j-1),(2,x,j-1),\ldots,(m,x,j-1)$ are all incoming edges of that vertex, so $A_{i,j} = \min(A_{k,j-1} + t(k,i,j-1))$ when $k = 1,2,3,\ldots,m$ for all $j \geq 2$

Considering $v_{i,1}$ we found that it only connects to the edge $(1,i,0)$. Therefore $A_{i,1} = t(1,i,0)$, completing the proof.

- Introduction
- Lexicographic Product
- Flow Network
- Our work
- **Results**

2. Maximum flow solution $B_{m,n}$

Theorem: If a graph G has unique maximum flow, removing an edge with highest time weight results in a graph that has its maximum flow pattern the same as the original graph.

Proof: Let N be the amount of maximum flow in the original graph. Assume the contrary that there exist some edges that its flow increases in the resulting graph. It is obvious that the total flow in the resulting graph cannot exceed N, so there are some free spaces allowing more flow to passes through to obtain the original N flow. That makes the maximum flow in the original graph not unique, contradicting the assumption.

3. Bounds on maximum efficiency

Theorem: For a maximum flow f in $B_{m,n}$,

$$1+\frac{\min(T(f))}{\max(T(f))}\le eff(f)\le 2$$

Proof:

Since $$eff(f)=\frac{|f|}{\max|f|}+\frac{\min(T(f))}{T(f)}$$ for any flow f.

Observe that $0\le eff_{\max imum\_flow}\le eff(f)\le 2$.

Considering the maximum flow, $|f|=\max|f|$ and obviously $T(f)\le\max(T(f))$.

Therefore $$\frac{\max|f|}{\max|f|}+\frac{\min(T(f))}{\max(T(f))}\le eff_{\max imum\_flow}$$

$$1+\frac{\min(T(f))}{\max(T(f))}\le eff(f)\le 2 .$$

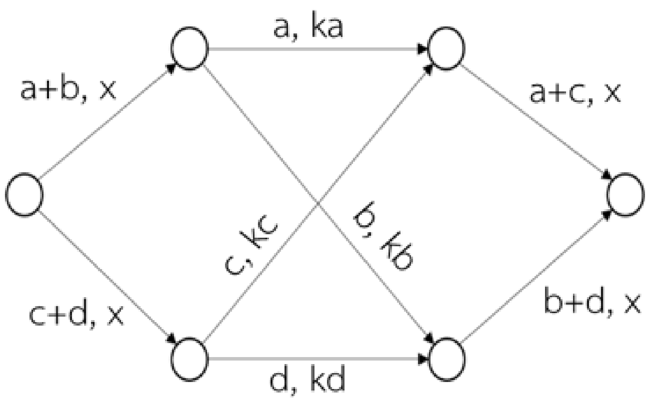
For a maximum flow f.

4. Maximum flow implies maximum efficiency

Theorem: In a graph $B_{2,2}$ with a condition that

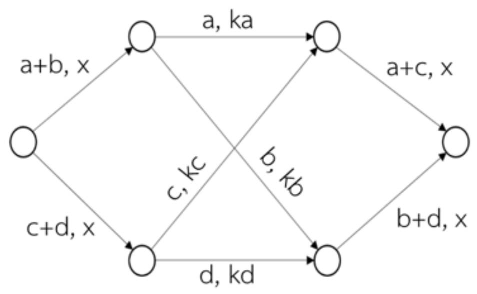
- $t(1,1,1)=a$, $t(1,2,1)=b$, $t(2,1,1)=c$ และ $t(2,2,1)=d$
- $c(1,1,1)=ka$, $c(1,2,1)=kb$, $c(2,1,1)=kc$ c และ $c(2,2,1)=kd$
- $t(1,1,0)\geq a+b$, $t(1,2,0)\geq c+d$, $t(1,1,2)\geq a+c$ และ $t(2,1,2)\geq b+d$
- $c(1,1,0)=c(1,2,0)=c(1,1,2)=c(2,1,2)=x$ เมื่อ $x=\dfrac{k(a+b+c+d)}{4}$

when $k,a,b,c,d\in\mathbb{R}^{+}$
We get that a solution which gives maximum flow also gives maximum efficiency.



4. Maximum flow implies maximum efficiency

Proof: Consider closely, there are four types of flow which can probably give the maximum flow.



Case 1 flow=a time=ka+2x

Case 2 flow=b time=kb+2x

Case 3 flow=c time=kc+2x

Case 4 flow=d time=kd+2x

Without loss of generality assume that $a \le b \le c \le d$

Case 1 time=ka+2x

We get that $eff_1 = \dfrac{a}{k_1} + \dfrac{k_2}{2x+ka}$ ....(1)

When $k_1 = a+b+c+d$ และ $k_2 = 2x+ka$

Case 2 time=kb+2x

We get that $eff_2 = \dfrac{a+b}{k_1} + \dfrac{k_2}{2x+kb}$ ....(2)

4. Maximum flow implies maximum efficiency

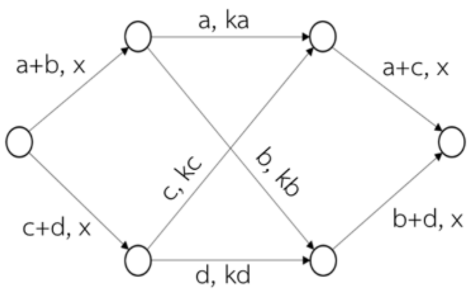
Case 3 time=kc+2x

We get that $$eff_3 = \frac{a+b+c}{k_1} + \frac{k_2}{2x+kc} \qquad \text{..........(3)}$$

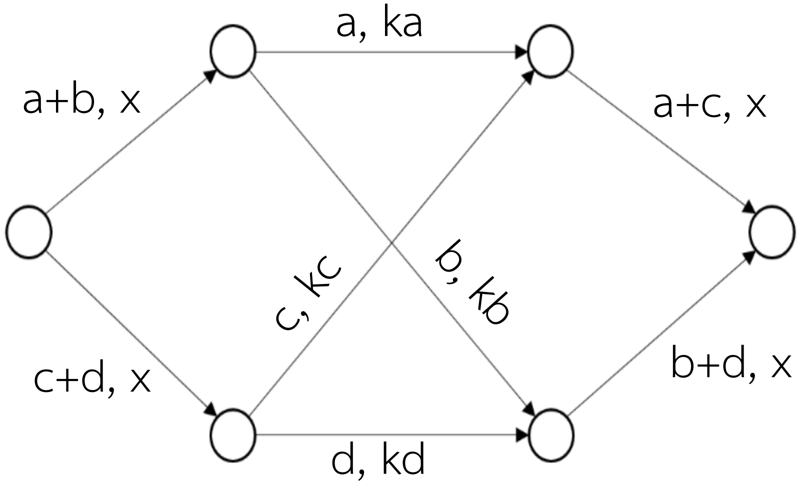
Case 4 time=kd+2x

We get that $$eff_4 = \frac{a+b+c+d}{k_1} + \frac{k_2}{2x+kd} \qquad \text{..........(4)}$$

We are left proving that $eff_4 > eff_1$, $eff_4 - eff_2 > 0$ and $eff_4 - eff_3 > 0$



4. Maximum flow implies maximum efficiency



When x= k*(a+b+c+d)/4

Consider eff 4 – eff 1 $= \frac{a+b+c+d}{k_1} + \frac{k_2}{2x+kd} = 1 + \frac{2x+ka}{2x+kd} > 1 + \frac{ka}{kd}$

$$> 1 + \frac{a}{d}$$

$$> 1 + \frac{a}{a+b+c+d}$$

$$> 1 + \frac{a}{k_1}$$

$$> \frac{k_2}{2x+ka} + \frac{a}{k_1} = \text{eff 1}$$

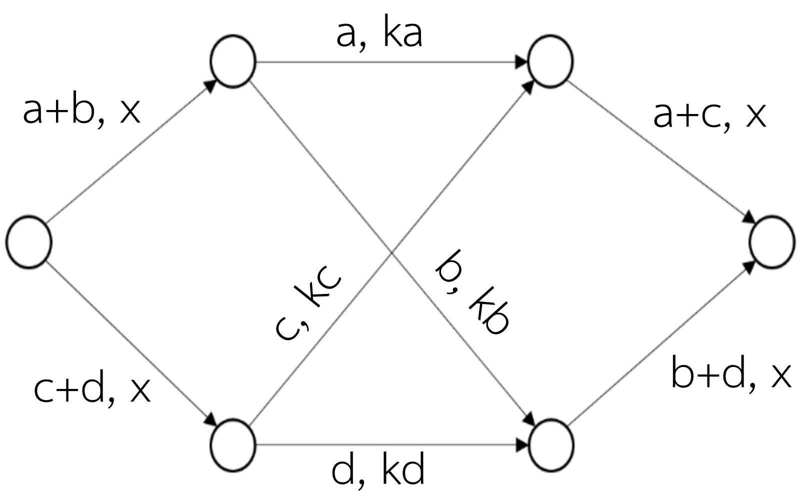
Consider eff 4 – eff 2 $= \left(\frac{a+b+c+d}{k_1} + \frac{k_2}{2x+kd}\right) - \left(\frac{a+b}{k_1} + \frac{k_2}{2x+kb}\right)$

$$= \frac{c+d}{k_1} - \frac{2(d-b)(3a+b+c+d)}{(a+3b+c+d)(a+b+c+3d)}$$

$$= \left(6a^2 b + a^2 c - 5a^2 d + 8ab^2 + 12abc + 4abd + 2ac^2 - 2acd - 4ad^2 + 2b^3 + 7b^2 c + 5b^2 d + 6bc^2 + 14bcd + 8bd^2 + c^3 + 3c^2 d + 3cd^2 + d^3\right) / ((a+b+c+d)(a+3b+c+d)(a+b+c+3d))$$

> 0

4. Maximum flow implies maximum efficiency



When x= k*(a+b+c+d)/4

Consider eff 4 – eff 3

$$= \left(\frac{a+b+c+d}{k_1} + \frac{k_2}{2x+kd}\right) - \left(\frac{a+b+c}{k_1} + \frac{k_2}{2x+kb}\right)$$

$$= \frac{d}{k_1} - \frac{2(d-c)(3a+b+c+d)}{(a+b+3c+d)(a+b+c+3d)}$$

$$= \left(\left(6a^2c - 5a^2d + 8abc - 6abd + 8ac^2 + 4acd - 4ad^2 + 2b^2c - b^2d + 4bc^2 + 4bcd + 2c^3 + 5c^2d + 8cd^2 + d^3\right) / \left((a+b+c+d)(a+b+3c+d)(a+b+c+3d)\right)\right)$$

> 0

Therefore, eff 4 > eff 1, eff 4 > eff 2, eff 4 > eff 3 for all positive real number x.

Meaning that maximum flow implies maximum efficiency.

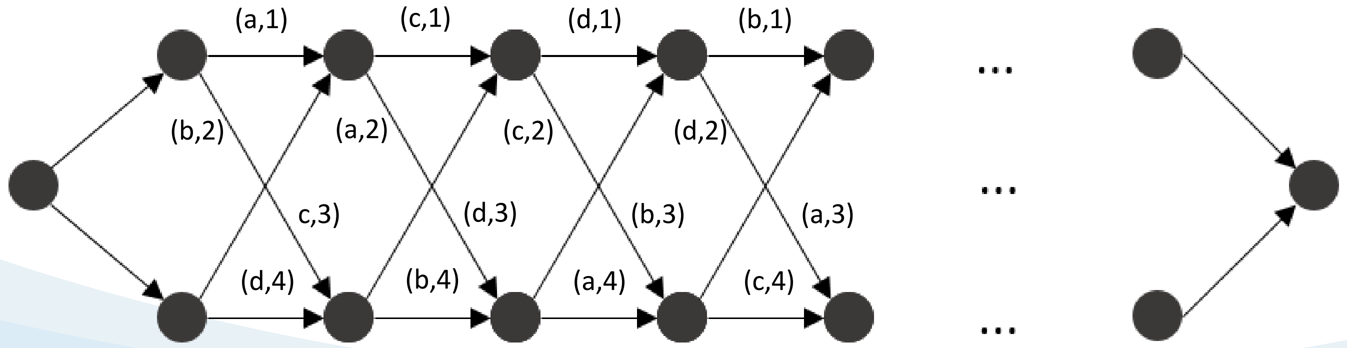
4. Maximum flow implies maximum efficiency

Theorem: Consider a graph $B_{2,4k+1}$ which obeys the following constraints.

- $c(1,1,0) \geq a+b$, $c(1,2,0) \geq c+d$, $c(1,1,4k+1) \geq a+c$ and $c(2,1,4k+1) \geq b+d$
- $t(1,1,0) = t(1,2,0) = t(1,1,4k+1) = t(2,1,4k+1) = 0$
- $c(1,1,4m+1) = c(1,2,4m+2) = c(2,2,4m+3) = c(2,1,4m+4) = a$ when $m = 1,2,...,k-1$
- $4k+1$ when $m = 1,2,...,k-1$
- $c(2,1,4m+1) = c(1,1,4m+2) = c(1,2,4m+3) = c(2,2,4m+4) = c$ when $m = 1,2,...,k-1$
- $c(2,2,4m+1) = c(2,1,4m+2) = c(1,1,4m+3) = c(1,2,4m+4) = d$ when $m = 1,2,...,k-1$
- $t(1,1,m) = 1, t(1,2,m) = 2, t(2,1,m) = 3, t(2,2,m) = 4$ เมื่อ $m = 1,2,...,k-1$

when $k,a,b,c,d \in \mathbb{R}^{+}$ and $a \leq b \leq c \leq d$.

Then the solution which gives maximum flow also gives maximum efficiency.

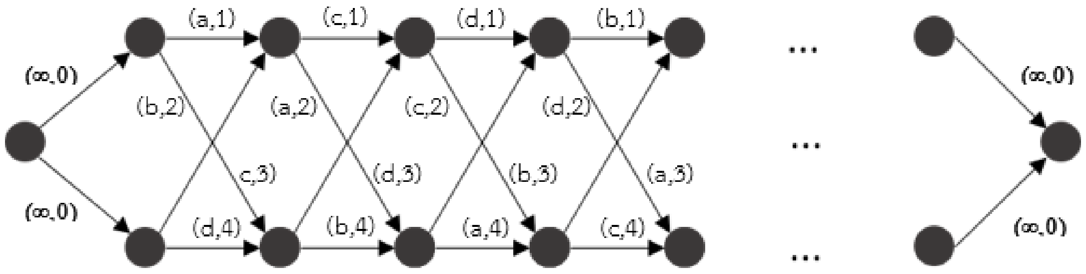


4. Maximum flow implies maximum efficiency

Let $\max|f| = M = a+b+c+d$

$\min(T(f)) = t = 4k$

So $eff = \dfrac{flow}{M} + \dfrac{t}{time}$



Again, we classify the flow into **7 cases**, as shown in the following table.

	time	*flow*	$\max(eff)$ of that case
1	16	$a+b+c+d$	$\frac{5}{4}$
2	15,14	$a+b+d$	$\frac{3a+4}{4a+6}+\frac{5}{4}$
3	13	$2a+d$	$\frac{3a+3}{4a+6}+\frac{4}{13}$
4	12	$a+b+c$	$\frac{3a+3}{4a+6}+\frac{4}{12}$
5	11,10	$a+c$	$\frac{2a+2}{4a+6}+\frac{4}{10}$
6	9,8,7	b	$\frac{a+1}{4a+6}+\frac{4}{7}$
7	6,5,4	a	$\frac{a}{4a+6}+1$

4. Maximum flow implies maximum efficiency

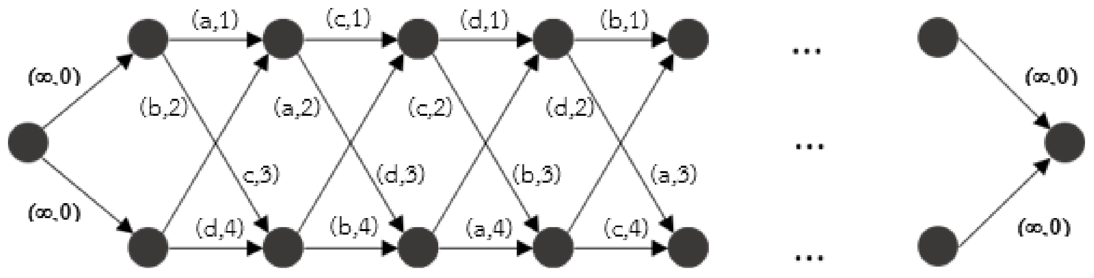
We can classify these cases further into 4 types relative to the number of paths used.

Type 1: 1

Type 2: 2,3,4

Type 3: 5

Type 4: 6,7



For $a \to \infty$

In type 1, we get $\lim_{a\to\infty} \frac{flow}{M} = 1$.

In type 2, we get $\lim_{a\to\infty} \frac{flow}{M} = \frac{3}{4}$.

In type 3, we get $\lim_{a\to\infty} \frac{flow}{M} = \frac{1}{2}$.

In type 4, we get $\lim_{a\to\infty} \frac{flow}{M} = \frac{1}{4}$.

Observe that in type 1, the pattern 1 gives the most eff, which is equal to $\frac{5}{4}$.

In type 2, the pattern 4 gives the most eff, which is equal to $\lim_{a\to\infty} eff_4 = \frac{13}{12}$.

In type 3, the pattern 5 gives the most eff, which is equal to $\lim_{a\to\infty} eff_5 = \frac{9}{10}$.

In type 4, the pattern 7 gives the most eff, which is equal to $\lim_{a\to\infty} eff_7 = \frac{5}{4}$.

Therefore, the flow pattern which gives the most eff is pattern 7, which is also the maximum flow pattern. Also, the flow with $(\min(T(f)))$ gives the second most efficiency.

Further improvement

1. Finding sharper lower and upper bound
2. Coding of program to extend the result of “maximum flow implies maximum efficiency”
3. Finding an explicit algorithm on finding maximum-efficiency flow on $B_{m,n}$
4. Implementation of the study into more real situations

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