MATS 2019. Senior league

Problem 1. Gavrila decided to weight his textbook on maths. He had no scales, but he had weights of 200 g, light ruler without label near its edges, a pencil, and weightless threads. Gavrila hung the textbook to one end of the ruler, one weight to the other end, and equilibrate the ruler at the pencil. Then, he added the second weight to the first one. To restore the equilibrium, he moved the pencil by 3 cm. After adding the third weight to the first and the second ones, and moving the pencil by 2 cm in the same direction, the equilibrium appeared again. What is the weight of the textbook?

Solution. Let M be a mass of the textbook, m be a mass of the weight. Values of the first and the second displacement of the pencil are x and y. The conditions of equilibrium of the ruler for three cases mentioned in the problem text read

$$Ml_1 = ml_2$$
$$M(l_1 + x) = 2m(l_2 - x)$$
$$M(l_1 + x + y) = 3m(l_2 - x - y)$$

Multiplying the first equation by 2 and 3 and subtracting it from the second and the third, respectively, we obtain

$$Mx - Ml_1 = -2mx$$
$$M(x + y) - 2Ml_1 = -3m(x + y)$$

which gives

$$M = \frac{3y - x}{x - y} m = 600$$

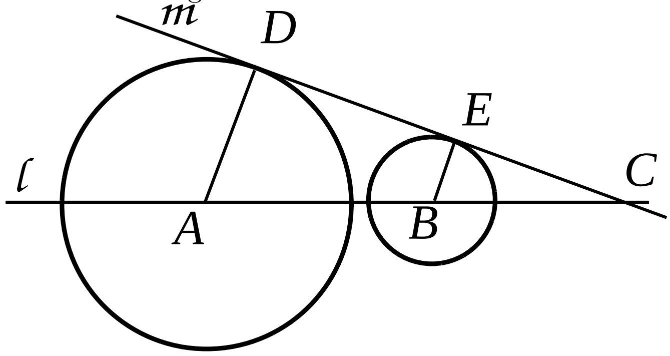
Answer: 600 g.

Problem 2. Points A and B lie on the straight line ↕, length of $AB = L$. At the initial moment, a circle appears at the point A. Initially, the radius of this circle is zero; later, it raises with speed a, the center is fixed. At the moment when the radius of the circle is R ($R < L$), the other circle appears at point B. Again, its radius rises from zero with the speed of a, and the center is fixed. These circles have a common tangent ⇳, points A and B lie on the same side from ⇳. What is the velocity of the point of intersection of l and m?

Solution. The triangles ADC and BEC (see picture) are formed by segments of tangent, line ↕ and radii AD and BE to the tangent points and the moment T. These triangles are similar, and we have

$$\frac{aT}{aT - R} = \frac{L + x}{x},$$

where x is the length of BC.



This equation gives $x = (aT - L)L/R$. And the velocity of the point C is aL/R

Note, that this problem models the sound propagation from the point source moving with supersonic velocity.

Answer: aL/R.

Problem 3. A fountain pump sucks water into a short vertical tube and pushes out the vertical jet. What is the factor of the pump power increasing to increase the jet height by the factor of 4? Justify your answer (losses for friction are negligible, the pump efficiency does not depend on its power).

Solution. As the friction losses are negligible, the jet height is proportional to v^2, v being the jet velocity at the pump outlet: $h \sim v^2$.

If the velocity v increases, the flow rate Q (mass per unit time) increases proportionally: $Q \sim v$.

The pump power is proportional to the kinetic energy production rate, i.e., product of flow rate to the quare of velocity:

$$P \sim Qv^2 \sim v^3 \sim h^{3/2}.$$

As h increases by the factor of 4, the power increasing factor is 8.

Answer: by the factor of 8.