Answers and solutions
Math Around Us
Senior League. 2019/10/29

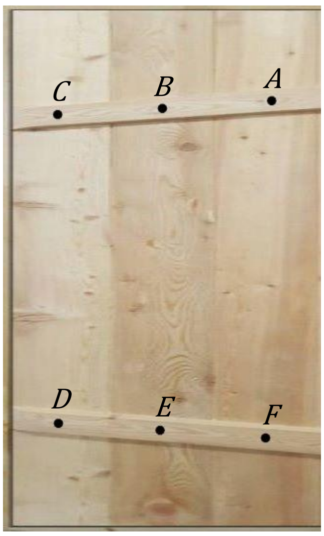
1. The panel of a digital code lock to a door has four buttons labeled 1, 2, 4, 8. To open the door, one must press three buttons in a certain order — this ordered set (unique) of three digits is called the access code. Imagine you have forgotten the code. You can press the buttons in any order and as many times as you need. As soon as the access code appears in the sequence you have dialed, a light on the panel will come on and the door will open itself. What is the shortest sequence you must dial without knowing the access code to ensure that the door will open if you know that the code can consist:
    a) of three different digits, but there is no digit 8 among them;
    b) of any three digits, no two of which are the same;
    c) of any digits, not necessarily different?

Answer: a) 9; b) $26 = 4 \cdot 3 \cdot 2 + 2$; c) $66 = 4^3 + 2$.
Solution. a) The desired chain of numbers, for example, is as follows: 124121421, and the shorter length is not enough (if the numbers were only $3 \cdot 2 \cdot 1 + 2 = 8$, then all of them would be cyclic permutations of some one triple of numbers).
b) Consider a directed graph whose vertices are ordered pairs (a, b) of various digits from a given set, and the edges are the allowed triples (a, b, c) that are formed by joining (with the arrow $\to$) vertex (b, c) to the right of the vertex (a, b). There are only $4 \cdot 3 = 12$ vertices and $4 \cdot 3 \cdot 2 = 24$ edges in this graph: each vertex has 2 outgoing edges and 2 incoming edges. This graph is connected, because from any vertex to any you can go along the edges, which the chain demonstrates

$$(a, b) \to (b, s) \to (s, d) \to (d, b) \to (b, a) \to (a, c).$$

Therefore, this graph has an Euler cycle that runs exactly once along each edge, which means that it sweeps out all possible triples (lock codes), and it sets the desired chain of digits (there are 2 more of them than edges, since the initial vertex adds two initial numbers).
c) Consider a similar graph in which all pairs (vertices, there are 16 ones) and all triples (edges, there are 64 ones) are allowed: each vertex has 4 outgoing and 4 incoming edges (it is possible that 2 edges form a loop), and this graph is connected. It also has an Euler cycle that sets the desired chain of digits.

2. A new door to a sauna is made of three vertical wooden boards fastened together with two crosswise bars, not necessarily horizontal, attached to the boards by bolts A, B, C, D, E, and F so that $AB = BC = DE = EF = 1$, $AF = 3$ and $CD = a$. Over time, the boards dry out and shrink in width, resulting in gaps between them which allow them to slightly rotate around the bolts. For what values of $a \in [2,4]$ the door will not become deformed over time (under the gravity force)? Will the segment of the middle board between B and E be in tension or contraction under the attempted deformation?



Answer: for any $a \neq 3$; it suffers a compression load.

Solution. Since BE is the «midline» of the quadrangle $ACDF$, then $2\overrightarrow{BE} = \overrightarrow{AF} + \overrightarrow{CD}$. Moreover, at the very beginning we have $AF \parallel CD$, therefore $2BE = AF + CD$. If the parallelism $AF \parallel CD$ is violated, then we obtain $2BE < AF + CD$, i.e. the BE segment is compressed, which is impossible. If the parallelism is not broken, then the quadrangle $ACDF$ will remain a trapezoid or a parallelogram with the original lengths of the sides: in the first case, it cannot be deformed, and in the second one it can.

3. Opening a water tap, each of us has observed that the stream of water flowing from the tap is getting narrower. Find the formula for the thickness $d = d(h)$ of the stream in terms of the distance h from the outlet of the tap. If needed, assume that the diameter of the tap hole is d_0 and the initial velocity of the stream is v_0. In which part of the jet: upper or lower, its thickness decreases faster with increasing h?

Answer: $d(h) = \sqrt[4]{\frac{v_0^2}{v_0^2+2gh}} \cdot d_0$, where g is the acceleration of gravity.

Solution. From the formulas $v = v_0 + gt$, $h = v_0 t + \frac{gt^2}{2}$ (for speed and path when accelerating gravity) and $v_0 d_0^2 = vd^2$ (constant fluid flow), excluding v, we find $v_0 d_0^2 = (v_0 + gt)d^2$, whence we express $t = \frac{v_0}{g}\left(\frac{d_0^2}{d^2} - 1\right)$, and then, excluding t, we find

$$h = \frac{v_0^2}{g}\left(\frac{d_0^2}{d^2} - 1\right) + \frac{v_0^2}{2g}\left(\frac{d_0^2}{d^2} - 1\right)^2 = \frac{v_0^2}{2g}\left(\frac{d_0^4}{d^4} - 1\right),$$

whence, expressing d through h, we get the answer.
From the obtained answer of the form $d(h) = \frac{c_1}{(c_2+h)^{1/4}}$ (where $c_1, c_2 > 0$) we find the derivative $d'(h) = -\frac{1}{4}\frac{c_1}{(c_2+h)^{5/4}}$ — it is negative, but decreases in absolute value with increasing h, i.e. the thickness of the jet at the top decreases faster than at the bottom.