1 Team contest

Senior League

1. (3) Does there exist an integer with decimal notation consisting of digits 5, 6, 8, 9 taken 2017 times each, that is divisible by some integer with decimal notation consisting of digits 1, 2, 3, 4 taken 2017 times each?
(inspired by Canadian Mathematical olympiad 2011)

2. (3) Prove that there exist infinitely many non-constant integer arithmetic progressions with 2016 terms, such that for each progression the product of its terms is a perfect 2017th power. *(Greece National Olympiad 1998)*

3. (4) Find all polynomials $P(x)$ with real coefficients which satisfy the condition

$$(x^2 - 6x + 8)P(x) = (x^2 + 2x)P(x - 2)$$

for all real x. *(Greece National Olympiad 2014)*

4. (4) Given a triangle ABC, a line tangent to its circumcircle is constructed through the point A. This line intersects the line BC at point P. Let Q and R be the reflections of P about lines AB and AC respectively. Prove that BC is perpendicular to QR. *(Japan Mathematical Olympiad Finals 2012)*

5. (5) A card deck consists of $2^n - 1$ cards. It is shuffled in the following way: top 2^{n-1} and bottom $2^{n-1} - 1$ cards are mixed by alternating them, while respecting order in each part (so the topmost card remains in its place). After how many such operations the deck will return to its original state? *(V. V. Novikov)*

6. (6) Let p be an odd prime number. Let a_k be the number of divisors d of the number $kp + 1$ satisfying $k \leqslant d \leqslant p$. Find the value of $a_1 + a_2 + \ldots + a_{p-1}$.
(Japan Mathematical Olympiad Finals 2016)

7. (7) On a 2018×2018 square grid, a piece can move between squares sharing a side. The piece begins in some square of the grid and enters each square exactly once, ending in the square where it started. What is the least k such that the piece can move in such a way that it enters each column and each row at most k times?
(Canada National Olympiad 2015)

8. (9) Let $x_1, \ldots, x_{100}$ be nonnegative real numbers such that $x_i + x_{i+1} + x_{i+2} \leqslant 1$ for all $i = 1, \ldots, 100$ (we put $x_{101} = x_1$ and $x_{102} = x_2$). Find the maximal possible value of the sum

$$S = \sum_{i=1}^{100} x_i x_{i+2} .$$

(Shortlist 2010)

9. (9) We are given a circle ω with center O and radius R. Two points A and B lie on the circle so that $R < AB < 2R$. The circle ω_1 with center A and radius less than R intersects the circle ω at points C and D, of which C belongs to the shorter arc AB. Two lines through the point B are tangent to the circle ω_1 at points E and F, in such way that E lies outside of the circle ω. Let M be the intersection of the lines EC and DF. Show that the quadrilateral $BCFM$ is cyclic.

(Greece National Olympiad 2014)

10. (10) Given an integer greater than 2017, prove that it can be decomposed into a sum of non-one positive integers whose product is the factorial of some positive integer. *(A. F. Nazmutdinov inspired by Tournament of Towns, oral round 2012)*