Regatta

Senior League

First Round

1a. Numbers $\sin(17^\circ)$ and $\sin(73^\circ)$ are roots of the quadratic trinomial $ax^2 + bx + c$. Prove that $a^2 + 2ac = b^2$.

1g. AD is a diameter of the circumcircle of the quadrilateral $ABCD$. Point E is symmetric to the point A with respect to the midpoint of BC. Prove that $DE \perp BC$.

1c. Sasha marked some cells of a table 8×13 so that there was and odd number of marked cells in every square 2×2. After that, he marked some more cells so that there was an even number of marked cells in every square 2×2. What is the minimum possible total number of marked cells?

Second Round

2a. Prove that for every positive integer n the value of the expression

$$\left[\frac{n}{1}\right] + \left[\frac{n}{2}\right] + \ldots + \left[\frac{n}{n}\right] + [\sqrt{n}]$$

is even.

2g. Incenter I of the acute triangle ABC lies on the bisector of an acute angle between the altitudes AA_1 and CC_1. Bisector of the angle B intersects the opposite side of the triangle at the point L. Prove that points A_1, I, L, C lie on one circle.

2c. Numbers from 1 to 50 are written on 50 cards, each number appears once. Kostya and Vitalik take turns at picking a card, until all cards are picked. Kostya moves first and wants a total sum of numbers on his cards to be divisible by 25. Vitalik wants to thwart him. Will Kostya be able to achieve his goal?

Third Round

3a. Find the largest positive integer n such that for every its prime divisor p number n is divisible by $p - 1$, but not by p^2.

3g. Given point P inside the triangle ABC such that points symmetrical to P with respect to the midpoint of BC and to the bisector of angle A lie on one line with the point A. Prove that projections of the point P on the sides AB and AC are equidistant from the midpoint of BC.

3c. A country has 100 cities. It is required to connect some pairs of cities with flights so that every city will be reachable from every other with a sequence of flights, and for every four cities A, B, C, D for which there are flights AB, BC, CD, there will also be AD. How many ways there are to connect the cities?

Fourth Round

4a. Prove that

$$\sqrt{\frac{1962}{2018}} < \frac{1963}{1964} \cdot \frac{1965}{1966} \cdot \ldots \cdot \frac{2017}{2018} < \sqrt{\frac{1963}{2019}}.$$

4g. Point L on the bisector of the angle A of the triangle ABC is such that $\angle LBC = \angle LCA = \angle LAB$. Prove that the lengths of the sides of the triangle form a geometric progression.

4c. Either "+" or "−" is written in each cell of a table 2017×2017. In one move it is allowed to alter every sign in a chosen cross (cross is a union of a column and a row). Is it possible to obtain the table with "+" in its every cell for any initial placement of signs (after an arbitrary number of moves)?