The 9th Mathematical Multiathlon

31 October–5 November 2016, Moscow

Algebra and number theory

Senior League

1. Real numbers $x, y, z \neq 0$ are such that an equality
$$\frac{x^2+y^2-z^2}{2xy}+\frac{y^2+z^2-x^2}{2yz}+\frac{z^2+x^2-y^2}{2xz}=1$$
holds.
Find all possible values of
$$\left(\frac{x^2+y^2-z^2}{2xy}\right)^{2016}+\left(\frac{y^2+z^2-x^2}{2yz}\right)^{2016}+\left(\frac{z^2+x^2-y^2}{2xz}\right)^{2016}.$$

2. Let $x_1, x_2, \ldots, x_{2016}$ be positive numbers such that $x_1+\cdots+x_{2016}=1$.
Prove that
$$(1+x_1)(1+x_2)\cdot\ldots\cdot(1+x_{2016})<1+\frac{1}{1!}+\frac{1}{2!}\cdot\frac{2015}{2016}+\frac{1}{3!}\cdot\frac{2014}{2016}+\cdots+\frac{1}{2016!}\cdot\frac{1}{2016}$$

3. Find all integer solutions of the system
$$\begin{cases} abcd=c+199 \\ 51a^6+5c^2+20|b+d|=2016 \end{cases}$$

4. Let $P_n(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ is a polynomial with integer coefficients such that the equation $P_n(x)=2016$ has at least five integer solutions. Prove that equation $P_n(x)=2001$ does not have an integer solutions.