Regatta

Senior League

First Round

1a. Find all positive integers n that for some prime number p satisfy

$$p \leq \frac{n^2}{5} < p + 1 .$$

1g. Given triangle ABC with right angle C, find all points X on the hypotenuse AB such that $CX^2 = AX \cdot BX$.

1c. A five-digit number does not include zero in its decimal notation, and every two its digits are different and have no common divisors except 1. Count all such five-digit numbers.

Second Round

2a. Given two real numbers x and y satisfying

$$x^2 - 3xy + 2y^2 + x - y = 0 \qquad \text{and} \qquad x^2 - 2xy + y^2 - 5x + 7y = 0 ,$$

prove that they also satisfy

$$xy - 12x + 15y = 0 .$$

2g. Given triangle ABC, point O is the center of its excircle that is tangent to the side BC and extensions of the sides AB and AC. Prove that points A, B, C and the center of the circumcircle of triangle ABO are concyclic.

2c. In a 10-episode TV series no character appears in every episode. Given that every six episodes share a common character, find minimal possible number of characters in the series.

Third Round

3a. Real numbers x, y, z satisfy

$$x + y + z = 0 \qquad \text{and} \qquad x^2 + y^2 + z^2 = 6 .$$

What is the greatest possible value of $|(x - y)(y - z)(x - z)|$?

3g. Given isosceles triangle ABC ($AB = BC$), D is the midpoint of its base AC, E is the projection of D on the side BC, and F is the second intersection of circumcircle of the triangle ADB with the segment AE. Prove that the line BF bisects the segment DE.

3c. On a plane 2016 equilateral (maybe not equal) triangles are given. What is the largest number of pieces they can split the plane into?

Fourth Round

4a. Arbitrary function $f : \mathbb{R} \to \mathbb{R}$ is given such that it is not a constant. Prove that there are such real x and y that $f(x + y) < f(xy)$.

4g. Triangle ABC with angle bisector BL is given such that $BL = AB$. Point M on the extension of BL beyond point L is such that $\angle BAM + \angle BAL = 180^\circ$. Prove that $BM = BC$.

4c. 2016 balls are arranged in a circular sequence, each ball is one of 32 colours, with exactly 63 balls of each colour. Find minimal n such that it is always possible to find n consecutive balls that have at least 16 different colours among them.