Senior League

Team Contest

1. (3) Is it possible to label the 8 vertices of a cube with distinct numbers from 1 to 8 in such a way that all sums on edges are different?

2. (4) Find all positive integers x, y, z such that

$$x^4 + x^2 = 7^z y^2.$$

3. (6) Let AB be a diameter of circle ω. ℓ is the tangent line to ω at B. Given two points C, D on ℓ such that B is between C and D. E, F are the intersections of ω and AC, AD, respectively, and G, H are the intersections of ω and CF, DE, respectively. Prove that $AH = AG$.

4. (6) Each square of an 8×8 board can be coloured with white or black. Find the number of colourings of the board such that every 2×2 square contains exactly 2 black squares and 2 white squares.

5. (7) Prove that for all positive reals a, b, c, d we have

$$\frac{a+b+c+d}{abcd} \leq \frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} + \frac{1}{d^3}\,.$$

6. (7) Let ABC be a triangle such that it's circumcircle radius is equal to the radius of escribed circle relative to A. Suppose that the escribed circle with respect to A touches BC, AC, AB at M, N, L. Prove that O (center of circumcircle) is the orthocenter of MNL.

7. (8) Let G be a connected graph. A vertex v has degree d, an edge e is incident to v. Let N be a number of spanning trees in G, and let n be a number of spanning trees in G containing e. Prove that $n \geqslant N/d$.

8. (9) In acute triangle ABC angle B is greater than C. Let M be the midpoint of BC. D and E are the feet of the altitudes from C and B respectively. K and L are midpoints of ME and MD respectively. If the line KL intersects the line through A parallel to BC at point T, prove that $TA = TM$.

9. (10) Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy

$$f(x^3 + y^3) = x^2 f(x) + y f(y^2)$$

for all $x, y \in \mathbb{R}$.

Team oral contest. Time allowed: 240 minutes.
The maximum score for each problem is shown in parentheses. The maximum total score is 60.
An unsuccessful attempt to present a solution decreases the maximum score on that problem by 1.
Each participant can present solutions to at most 3 different problems.