Regatta

Senior League

First Round

1a. a, b and c are positive integers and $\dfrac{a\sqrt{3}+b}{b\sqrt{3}+c}$ is a rational number. Show that $\dfrac{a^2+b^2+c^2}{a+b+c}$ is an integer.

1g. P is any point inside a triangle ABC. The perimeter of the triangle $AB+BC+CA=2s$. Prove that $s < AP+BP+CP < 2s$.

1c. Is it possible to fill a 3×3 grid with each of the numbers $1, 2, \dots, 9$ once each such that the sum of any two numbers sharing a side is prime?

Second Round

2a. The integers $1, 2, \dots, n$ are written in order on a long slip of paper. The slip is then cut into five pieces, so that each piece consists of some (nonempty) consecutive set of integers. The averages of the numbers on the five slips are 1234.5, 345, 128, 19, and 9.5 in some order. Compute n.

2g. Let ABC be an acute-angled triangle in which $\angle ABC$ is the largest angle. Let O be its circumcentre. The perpendicular bisectors of BC and AB meet AC at X and Y respectively. The internal angle bisectors of $\angle AXB$ and $\angle BYC$ meet AB and BC at D and E respectively. Prove that BO is perpendicular to AC if DE is parallel to AC.

2c. Let $p_1, p_2, \dots, p_{30}$ be a permutation of the numbers $1, 2, \dots, 30$. For how many permutations does the equality

$$\sum_{k=1}^{30} |p_k - k| = 450$$

hold?

Third Round

3a. In a quadratic table with 2015 rows and 2015 columns is an odd number written in each field. For $1 \le i \le 2015$ is Z_i the sum of the numbers in the i-th row and for $1 \le j \le 2015$ is S_j the sum of the numbers in the j-th column. A is the product of all Z_i and B the product of all S_j. Show that $A + B \neq 0$.

3g. Let $\triangle ABC$ be an acute scalene triangle, and let N be the center of the circle which pass trough the feet of altitudes. Let D be the intersection of tangents to the circumcircle of $\triangle ABC$ at B and C. Prove that A, D and N are collinear iff $\angle BAC = 45^\circ$.

3c. A school has n students and some super classes are provided for them. Each student can participate in any number of classes that he/she wants. Every class has at least two students participating in it. We know that if two different classes have at least two common students, then the number of the students in the first of these two classes is different from the number of the students in the second one. Prove that the number of classes is not greater that $(n-1)^2$.

Fourth Round

4a. The coefficients a, b, c of a polynomial $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3 + ax^2 + bx + c$ are mutually distinct integers and different from zero. Furthermore, $f(a) = a^3$ and $f(b) = b^3$. Determine a, b and c.

4g. ABC is an acute angle triangle such that $AB > AC$ and $\angle BAC = 60^\circ$. Let's denote by O the center of the circumscribed circle of the triangle and H the intersection of altitudes of this triangle. Line OH intersects AB in point P and AC in point Q. Find the value of the ration $PO : HQ$.

4c. Coins are placed in a 20×15 grid. 2 coins are called "neighbors" if they occupy the same row or column and there are no coins in between them. If each coin is allowed at most 2 neighbors, what is the greatest amount of coins that may be placed in the grid?