The 8th Mathematical Multiathlon

2–7 November 2015, Moscow

Geometry

Senior League

1. Two congruent circles b and c touching each other are inscribed in the angles B and C of a square $ABCD$. A tangent to circle b is drawn from vertex A and a tangent to c is drawn from D (see Fig. 1). Prove that the circle inscribed in the triangle bounded by these tangents and the side AD is congruent to the given circles.

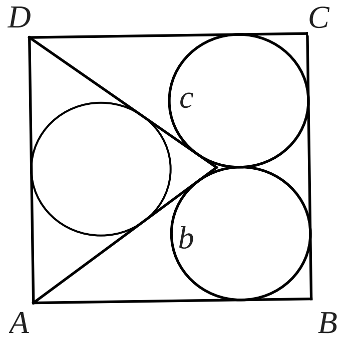

Fig. 1

2. (a) (3 points) Let $A_1A_2A_3A_4A_5$ be a pentagram, i.e., a closed self-intersecting five-edge polyline (see Fig. 2). Denote by B_1, B_2, …, B_5 the points of intersection of its nonadjacent edges: B_n is the intersection point of A_iA_{i+1} and A_kA_{k+1}, where the indices of all points are different and we assume A_6=A_1. Prove that

$A_1B_2 \cdot A_2B_3 \cdot A_3B_4 \cdot A_4B_5 \cdot A_5B_1 = B_1A_2 \cdot B_2A_3 \cdot B_3A_4 \cdot B_4A_5 \cdot B_5A_1$.

(b) (4 points) Prove that this equation holds for any polyline $A_1A_2A_3A_4A_5$ that has no parallel edges if we define B_n as the intersection points of the same edges as in (a) or their extensions.

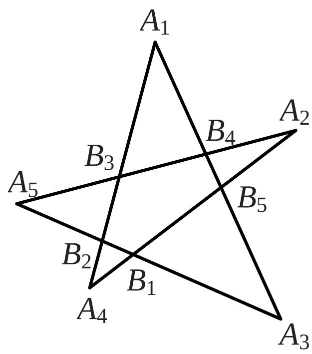

Fig. 2

3. Let n + 1 points, A_1, A_2, ..., A_n and B, be given in the plane such that no three of them are collinear. It is known that for any two points A_i and A_j one can find a third point A_k that completes a triangle $A_iA_jA_k$ containing the point B in its interior. Prove that n is odd.

4. On the sides BC and AC of a triangle ABC points B' and A', respectively, are taken. The circumcircle of the triangle ABC meets the line through C parallel to $A'B'$ in a point D (other than C). The circumcircle of the triangle $A'B'C$ meets the line through C parallel to AB in a point E (other than C). Prove that the lines AB, $A'B'$, and DE are concurrent.