MATS Olympiad 2025

Senior league

Problem 1

Pasha is walking through a store and wants to buy his favorite candies. He usually bought either a small box for $S_1 = 300$ rubles containing exactly $N_1 = 5$ candies, or a large box for $S_2 = 700$ rubles. He never counted how many candies are in the large box (N_2), but he is sure there are more than 20, since once he counted 20 pieces and there were still some left.

Today he noticed a medium box, larger than the small one but smaller than the large one. He has only 435 rubles with him, and the price tag on the box is smudged; help Pasha estimate the price of this box and determine whether he has enough money to buy it.

It is known that the manufacturer is fair, and the price of a box of candies is formed from the price of the candies and the price of the cardboard the box is made of, in proportion to their masses. The masses of the small and large boxes themselves without candies are known: they are $m_1 = 100$ g и $m_2 = 200$ g, respectively. The price tag on the medium box is smudged but not completely; Pasha can make out that its mass is $m_3 = 150$ g. Pasha is also confident that the medium box contains no more than 15 candies (N_3).

Solution:

The price of a candy box consists of the price for the mass of the candies and the price for the box itself. Let k_2 be the price in rubles per gram of cardboard (box material), – k_1 be the price in rubles per gram of candies, and let m_0 be the mass of one candy in grams; then we can write the system:

$$m_0 \cdot N_1 \cdot k_1 + k_2 \cdot m_1 = S_1,$$

$$m_0 \cdot N_2 \cdot k_1 + k_2 \cdot m_2 = S_2,$$

$$m_0 \cdot N_3 \cdot k_1 + k_2 \cdot m_3 = S_3.$$

From the first two equations, express k_2 and $A = m_0 \cdot k_1$, since these are two linear equations in two unknowns:

$$A = \frac{m_1 S_2 - m_2 S_1}{m_1 N_2 - m_2 N_1}, \qquad k_2 = \frac{N_2 S_1 - N_1 S_2}{m_1 N_2 - m_2 N_1}.$$

There are two solution paths, the first exact and the second not entirely rigorous.

1) We will estimate the value S_3.
Substituting A, k_2 into the equation for the medium box price yields:
$$S_3 = 50 \cdot \left[9 + \frac{2N_3 - 15}{N_2 - 10}\right].$$
Its minimization depends on the sign of the numerator (the denominator is always positive by the problem's conditions). If the numerator is negative, choose the smallest N_3 among those that make the numerator negative and minimize N_2 to make the negative fraction as large in magnitude as possible. If the numerator is positive, to minimize S_3 choose the smallest N_3 that makes it positive and choose N_2 as large as

possible. In this case, we get $N_3 = 6, N_2 = 20$ and $N_3 = 15, N_2 \to \infty$ or the second case, and the values of S_3 are 435 and 450 rubles, respectively.

2) Estimate each quantity. The value A has a fixed numerator, and only N_2 changes in the denominator; thus at the minimal $N_2 = 21$ it attains its maximum value, and for estimation we will use $N_2 = 20$. Above N_2 is not bounded by the problem statement, so formally the minimal value of A tends to 0 as $N_2 \to \infty$.
The value of k_2 in the numerator has N_2, so we substitute the numerical values of the other variables and convert the fraction:
$$k_2 = \frac{N_2 \cdot 300 - 5 \cdot 700}{100 \cdot N_2 - 200 \cdot 5} = \frac{3N_2 - 35}{N_2 - 10} = 3 - \frac{5}{N_2 - 10}$$
It is now clear that at minimal N_2 this value is minimal, and as $N_2 \to \infty$ it is maximal.
Therefore $A \in (0; 10)$, $k_2 \in (2.5; 3.0)$.
But we must account that A takes its minimal value when k_2 takes its maximal value and vice versa. For the estimate, we want the minimal price of the medium box; then take $N_3 = 6$. Thus
$$S_3 \in (435; 450).$$
It is important to note that these estimates give an interval with open endpoints, and Pasha has exactly 435 rubles in hand, hence he will not have enough money for the medium box. It remains to prove that among intermediate feasible cases (not every pair A, k_2 in these ranges is possible, since both depend on the single parameter N_2) there is no case that yields a smaller sum. In this case, it must be proven that there is no intermediate case that gives a smaller value. Without this, the solution cannot be considered strict, unlike the first option, in which we immediately obtained the minimums in both cases.

Problem 2

Vova forgot to defrost 2 kg of chicken that lies in the freezer at −18 °C. Mom will arrive in 30 minutes.

There is one quick way to defrost the chicken: put it in hot water (85°C) while continuously refreshing the water so that the temperature remains constant.

Vova has been in this situation before and knows that if the surface temperature exceeds 60°C, white spots will appear — his mom will immediately notice the changes (scalding).

It is required to estimate whether the chicken will defrost within 30 minutes, i.e., its center reaches 0 °C and its surface does not scald (does not reach 60 °C). Assume that inside the chicken there is a small but finite "center" region occupying 0.13% of the whole chicken, and treat the "surface" as a thin layer near the chicken surface occupying 0.33%. Heat spreads instantly within the chicken, but the center receives 0.1% of the total heat flux, and the surface 1%.

Chicken density: $\rho = 1000$ kg/m³.
Specific heat of chicken: $c = 2800$ J/(kg·K).
Convective heat transfer coefficient with water: h_ water= 40 W/(m²·K)
(Chicken gets heat from the water through the surface, the more temperature difference, the more heat flux)

Solution:

Estimate the chicken's surface area. Chicken volume: $V = \frac{m}{\rho} = \frac{2}{1000} = 0{,}002\ m^3$. Assuming it is a sphere, so its radius is

$$R = \left(\frac{3V}{4\pi}\right)^{1/3} \approx 0{,}078\ m.$$

Chicken surface area:

$$S = 4\pi R^2 \approx 0{,}077\ m.$$

Assume the heat-flux power is proportional to the surface area and the difference between the surface and ambient temperatures:

$$P = hS\Delta T.$$

Write equations for the center and surface temperatures. We know what fraction of the total heat flux P goes to each part and the mass of each part, so the heat needed to raise the temperature is

$$c \cdot m \cdot \big(T(t) - T(0)\big) = P \cdot k \cdot t,$$

where k – is the fraction of heat delivered to the part of mass m.

For the center and the surface:

$$m_c = m \cdot a_c, \qquad a_c = 0.0013, \qquad k_c = 0.001, \quad T_c = T_c(0) + \frac{hS}{cm} \cdot \Delta T \cdot k_c \cdot t \cdot \frac{1}{a_c},$$

$$m_s = m \cdot a_s, \qquad a_s = 0.0033, \qquad k_s = 0.01, \quad T_s = T_s(0) + \frac{hS}{cm} \cdot \Delta T \cdot k_s \cdot t \cdot \frac{1}{a_s}.$$

Note that ΔT cannot remain constant; at the very start it is 103 °C, but if the surface approaches 60 °C, then ΔT is only 25 °C.

From the equation, we can see that the difference between the desired temperature and the initial temperature is linearly dependent on time and ΔT.This means that the temperature in the center and on the surface will differ in a proportional manner for different ΔT, reaching specific absolute values at different times t. This means that if the surface of the chicken is not boiled in any case, then it will not be boiled in the ΔT that depends on time. Therefore, let's consider the worst-case scenario where the chicken is on the verge of being boiled and its surface has a temperature of 60 °C, thus ΔT is 25 °C.

Now compute the temperatures at the surface and at the center after 30 minutes:

$$T_c(30\ min) \approx 0.54\ °C, \qquad T_s(30\ min) \approx 56.8\ °C.$$

Therefore, over these 30 minutes the difference ΔT does not fall below 25 °C the surface does not exceed 60 °C, and the center surpasses 0 °C, so the chicken will defrost in time.

(Note: if the form in the solution differs from the sphere, this solution also will be accepted, if all calculations are correct, notice that answer may differ from the answer below).

Problem 3

100 mg of spoiled juice entered the body. In the first 30 minutes, the concentration in the blood decreases by 5% every minute, then the elimination rate halves and this continues every next half-hour.

There are two options: stay and wait for natural elimination at home or immediately go to the pharmacy (one-way trip takes 15 minutes). While walking, elimination slows by 15% due to tiredness every 5 minutes (the effect manifests only after minute 5; similarly, upon returning home the tiredness effect immediately disappears at minute 30). An adsorbent from the pharmacy is taken only at home after returning and increases the elimination rate by 100%, but not immediately: it reaches full effect in 5 minutes.

The safe level is 6 mg; if it is not reached within 1.5 hours after ingestion, stomach pain will occur; choose a strategy.

Solution:

Let the initial concentration be C_0. Each subsequent i-th minute the concentration decreases by 5%:

$$C_{i+1} = C_i \cdot 0.95.$$

Write out the first few terms of this sequence (such sequences are called recurrent) and express each through all the previous ones:

$$C_3 = C_2 \cdot 0.95^1 = C_1 \cdot 0.95 \cdot 0.95 = C_1 \cdot 0.95^2 = C_0 \cdot 0.95^3.$$

Note that each i-th term is expressed via the zeroth term and a power of 0.95, so while the elimination rate v remains constant (here v=0.05), the formula holds:

$$C_i = C_0 \cdot (1 - v)^i.$$

After half an hour the elimination rate halves, the new $v = 0.025$, accordingly one may take C_{30} as the new C_0 and repeat as many times as needed.

Compute the concentration after half an hour, one hour, and one and a half hours, respectively:

$$C_{30} = C_0 \cdot 0.95^{30},$$

$$C_{60} = C_{30} \cdot 0.975^{30},$$

$$C_{90} = C_{60} \cdot 0.9875^{30}.$$

Thus after 30 minutes the concentration will be $\approx 21.5\,mg$, after an hour $\approx 10.0\,mg$, after an hour and a half $\approx 6.9\,mg$; therefore, this option is unsuitable.

Consider the option of going to the pharmacy for the medicine. The base elimination rate is 5% per minute; while walking it slows stepwise every 25 minutes, so in the first 5 minutes the previous formula holds:

$$C_{i+1} = C_i \cdot 0.95.$$

Then at minute 5 the first walking- tiredness effect occurs; the law for the next minute changes to

$$C_{i+1} = C_i \cdot (1 - 0.05 \cdot 0.75).$$

Five minutes later the law becomes

$$C_{i+1} = C_i \cdot (1 - 0.05 \cdot 0.75^2).$$

And so on, four more times at minutes 10, 15, 20, and 25 (nothing special happens at minute 30, since we assume tiredness instantly vanishes upon arriving home), because the trip is 15 minutes to the pharmacy and 15 minutes back.

For this stage, make a worst-case estimate: the worst case is if over the entire 30-minute interval the fatigue effect is maximal, as at minute 25; then for the first 30 minutes the bound holds

$$C_{30} < C_0 \cdot (1 - 0.05 \cdot 0.85^5)^{30} \approx 51.0\, mg.$$

(Note: without the estimation, the accurate value would be 34.7 mg).

Next, the medicine is taken. Also remember that the rule "the rate halves every half hour" still applies, so $v_1 = 0.025$. For the medicine, one may assume the following law during the first 5 minutes, where τ is the time since taking the medicine (from 0 to 5 minutes):

$$v = v_1 \cdot \left(1 + 0.8 \cdot \frac{\tau}{5}\right).$$

During this period the concentration still decreases each minute, so again make a worst-case estimate — the worst case is when the medicine does not act at all during the first 5 minutes and only starts acting after 5 minutes; then for the first 5 minutes:

$$C_{i+1} = C_i \cdot 0.975.$$

Hence after 5 minutes the concentration equals

$$C_{35} = C_{30} \cdot 0.975^5 \approx 44.9.$$

After that, the elimination rate increases by 80% due to the medicine and becomes $v_2 = 0.025 \cdot 1.8 = 0.045$, and for the next 25 minutes the law holds

$$C_{i+1} = C_i \cdot (1 - 0.045).$$

Then

$$C_{60} = C_{35} \cdot 0.955^{25} \approx 12.5.$$

After which the rate halves again $v_2 = 0.045 \cdot 0.5 = 0.0225$ and after 30 minutes the concentration becomes

$$C_{90} = C_{60} \cdot 0.9775^{30} \approx 5.8.$$

(Note: if all computations were exact, without worst-case bounds, the final concentration after 1.5 hours would be ≈ 3.8*).*