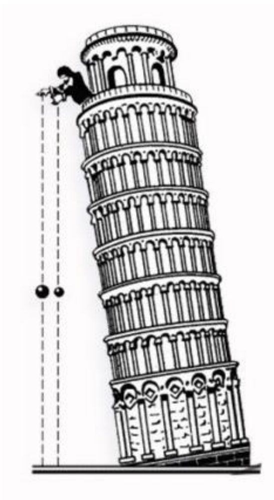
MATs Olympiad 2024.

Problem 1. Junior.

A small ball is released without initial velocity from a height of 20 m. It falls to the ground, then the ball is absolutely elastically reflected from it and in the absence of air resistance periodically moves between the ground and the starting point. Where (except for the starting point and the ground surface) does the ball go through at equal intervals of time? Determine the duration of this interval.



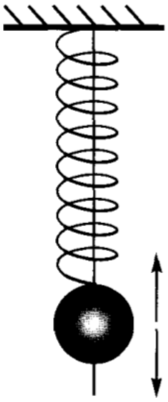
Solution:

The ball makes a periodic motion with period $T = 2\tau$, where $\tau = \sqrt{\frac{2H}{g}} = 2$ s. At each point between the ground surface and the starting point, the ball appears twice during the period: in upward and downward motion. The intervals between two consecutive appearances of the ball at a point will be constant if they are τ. This will be the case if the travel time from the starting point to the observation point is $\tau/2$, so it is at a distance of

$$d = \frac{g\tau^2}{8} = \frac{H}{4} = 5 \text{ м}$$

from the starting point, and hence at **a height $\boldsymbol{h = H - d = 15}$ m above the ground**. The time interval in this case, as noted, is, $\boldsymbol{\tau = \sqrt{\frac{2H}{g}} = 2}$ **s**

Problem 1. Senior.



A small ball with a lamp fixed on it hang on a spring and oscillates along a vertical ruler with centimetric scale. The lamp illuminates the mark of 10 cm once in 1 second and the mark of 5 cm once in 2 seconds.

a. Assuming that the light from the lamp goes within a narrow horizontal beam, find which marks are illuminated during the motion.
b. Let the mark be illuminated if the vertical distance between it and the lamp is smaller than 1 mm. Estimate the duration of each illumination of the mark of 10 cm.

For calculations, take π=3.1, 1/π≈0.32

Solution:

The vertical coordinate of the lamp is found from the law of harmonic oscillations

$$z = z_0 + A\cos\omega t$$

A point on the ruler is illuminated if its coordinate is the same as the coordinate of the lamp. By the problem statement, the roots of the equation

$$10 = z_0 + A\cos\omega t_1$$

form an arithmetic progression with a difference of 1 sec, and the roots of equation

$$5 = z_0 + A\cos\omega t_2$$

arithmetic progression with twice the difference. This situation is possible only if t_1, t_2 meet the conditions $\cos\omega t_1 = 0,\ \cos\omega t_2 = \pm 1, \frac{2\pi}{\omega} = 2\ s$. Hence, we determine the average position of the weight $z_0 = 10$ cm and amplitude $A = 5$ cm. So, the lamp will be near all divisions from 5 to 15 cm inclusive.

That is, firstly, there are points that are illuminated at equal intervals of time, for a spring pendulum there are only 3 such points: the unstretched position, and the extreme points, in which the velocity is 0. The extreme points are illuminated with a periodicity equal to the period of the pendulum, and the "middle" of its motion - the point at which the spring is in the unstretched position, and the velocity has maximum value, is illuminated twice per period in a time equal to half the period of the pendulum. Thus, except for the extreme points there is a single point that is illuminated for equal time intervals and in which the interval is smaller (1s < 2s), so the division of 10 cm is the point $z = z_0$, and the divisions of 5 and 15 cm are the extreme illuminated points on the ruler, and $T = 2$ s is the period.

A division of 10 cm is illuminated while the lamp travels a distance of 2 mm in the vicinity of the middle position. The velocity of the lamp at this time can be considered as maximum, because it changes only slightly, since the distance of 2 mm is much smaller than the amplitude. The magnitude of the velocity is determined by the amplitude:

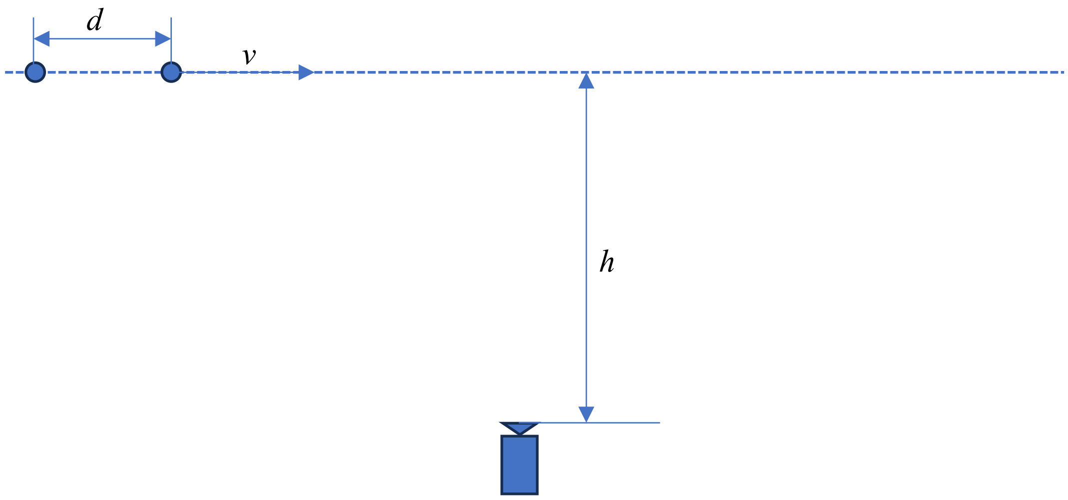
$$V_1 = A\omega = A \cdot \frac{2\pi}{T}$$

So, the time it takes lamp to cover a given distance:

$$\tau \approx \frac{d}{A}\frac{T}{2\pi} = \frac{2\cdot 10^{-3}\cdot 2}{5\cdot 10^{-2}\cdot 2\pi} = \frac{2}{5\pi}\cdot\frac{1}{10} = 0.04\frac{1}{\pi} \approx 0.0128\ s$$

Problem 2. Junior and Senior.

Two point sources of light slide at constant velocity *v* along the straight line located at the distance d from the observer that measures the light intensity from the sources with a small detectors directed straight to the sources at any instant. At the beginning of the observations, the sources are at the far left from the observer and at the end at the far right. The distance between the sources is constant and equals d. The maximal measured intensity of light from the second source turned out to be smaller than the maximal intensity of light from the first source. Meanwhile, at some instants, the intensity of light from the second source was larger than those from the first source. Determine the conditions when this could be observed, given the ratio of the maximum measured intensities is α<1. The light intensity at a given point is inversely proportional to the square of the distance to the source.



Solution:

By the statement of problem, the instantaneous intensities of the received signals are

$$I_1(t) = \frac{I_0}{h^2 + (vt)^2}, \qquad I_2(t) = \frac{\alpha I_0}{h^2 + (vt - d)^2}.$$

The constraint $I_2(t) > I_1(t)$ is equivalent to the inequality

$$(1-\alpha)v^2t^2 - 2vdt + d^2 + (1-\alpha)h^2 < 0,$$

The expression obtained on the left is a quadratic polynomial of t, the coefficient in front of the principal degree of t^2 is positive, so the inequality is satisfied between the roots of this parabola, this value is given by the formula (as the difference of the roots)

$$\frac{\sqrt{D}}{a}$$

Duration of time interval when this inequality is satisfied

$$\Delta t = \frac{2}{v(1-\alpha)}\sqrt{\alpha d^2 - h^2(1-\alpha)^2}.$$

The situation described in the condition is possible if the distance between the sources is sufficiently large: $d > \frac{h(1-\alpha)}{\sqrt{\alpha}}$.

Problem 3. Junior and Senior.

Metal pipes with hot water flowing inside are used for a room heating. During the tests, it was measured that the heat flux from the pipe per unit length is proportional to the pipe radius and insignificantly depends on the water temperature in the working range. When water goes through the 1 cm radius 6 meters long pipe and the water velocity is 1 m/s, its temperature at the inlet is 100°C, and drops down to 90°C at the outlet.

a. What is the water temperature at the outlet of the 2 cm radius 6 meters long pipe, if the water velocity is also 1 m/s and the temperature at the inlet is also 100°C?

b. Find a temperature at the outlet of the composite pipe that consists of two consecutive segments 3 meters long each. The radii of the segments are 1 cm and 2 cm, water velocity at the narrow segment is 1 m/s. Ehe temperature at the inlet is also 100°C.

Solution:

a. Let us write down the heat balance equation, which relates the difference between the input and output energy and the heat sink capacity, for pipes with radii R_1, R_2:

$$c\rho v\Delta t\pi R_1^2(T_0 - T_1) = \alpha R_1 L\Delta t \quad (1)$$

$$c\rho v\Delta t\pi R_2^2(T_0 - T_2) = \alpha R_2 L\Delta t$$

(Here it is assumed that the velocity at the inlet and outlet of the pipe is the same, so the same mass of liquid enters and exits, but which has lost 10 degrees of heat only due to heat exchange with the walls of the pipe, so the expression with heat capacity according to its definition is used). Standard notations are used for density, specific heat capacity, v is the velocity of the fluid, T_0 − is the temperature at the inlet of the pipe, T_1, T_2 – are the temperatures at the outlet. Dividing one equation by another we exclude α and find

$$\boldsymbol{T_2 = T_0 - \frac{R_1}{R_2}(T_0 - T_1) = 95°C}.$$

b. In the case of a composite pipe, as we pass from one segment to another, the velocity of the fluid changes so that the mass passing through each section is the same:

$$\rho v_1\Delta t\pi R_1^2 = \rho v_2\Delta t\pi R_2^2 = M$$

The quantity is also constant and is called the volume flow rate:

$$v_1\pi R_1^2 = v_2\pi R_2^2 = Q$$

Let us denote by T_* the temperature of the fluid at the junction of the segments. Then from the heat balance equation

$$c\rho Q\Delta t(T_0 - T_*) = \frac{\alpha R_1 L\Delta t}{2}$$

$$c\rho Q\Delta t(T_* - T_3) = \frac{\alpha R_2 L\Delta t}{2}$$

Using the first equation (1) and considering that the first pipe before articulation has the same velocity as in (1), we obtain, by dividing the first equation above by the first equation from (1), that the temperature at the outlet of the first pipe will be $\boldsymbol{T_*} = \frac{\boldsymbol{T_0+T_1}}{\boldsymbol{2}} = \boldsymbol{95^\circ C}$.

Using now the expression for $\boldsymbol{T_*}$ and similarly dividing the first equation by the second, we obtain the expression for $\boldsymbol{T_3}$:

$$\boldsymbol{T_3} = \frac{\boldsymbol{T_0+T_1}}{\boldsymbol{2}} - \frac{\boldsymbol{R_2}}{\boldsymbol{R_1}} \cdot \frac{\boldsymbol{T_0-T_1}}{\boldsymbol{2}} = \boldsymbol{85^\circ C}$$