Math Around Us
2024/10/29

Problem 1

Winnie the Pooh and Piglet play the following game: Winnie the Pooh rolls a **cube** with the numbers 1, 2, ..., 6 on its faces and gets the number of points that appears on its top side, while Piglet tosses **three coins** and gets the sum of three numbers: **two points** for each "heads" and $\boldsymbol{r}$ **points** for each "tails". The winner is the one whose resulting score is higher.

1. What is Piglet's probability of **winning** for:

A) $\boldsymbol{r = 0}$ if in case of **equal scores** Piglet **wins**;

B) $\boldsymbol{r = 1}$ if in case of **equal scores** Piglet **loses**?

2. For what values of $\boldsymbol{r} \in (\boldsymbol{0}, \boldsymbol{1})$, the probabilities of winning and losing for Piglet **are the same** if the case of **equal scores** is regarded as a **draw** (i.e. no one wins or loses)?

Answer: **1.A)** $\frac{1}{2}$; **1.B)** $\frac{7}{12}$; **2.** $r = \frac{1}{3}$.

Solution

1.A. In this case, the probability p_+ that Piglet wins is the sum of probabilities of the four events defined by the number of "heads," in which Piglet gets, respectively,

$$3 \cdot 2 = 6,\ \ 2 \cdot 2 = 4,\ \ 1 \cdot 2 = 2, \text{ or } 0 \cdot 2 = 0 \text{ points},$$

and Winnie the Pooh gets the same or smaller number of points. This gives the following expression:

$$p_+ = \frac{1}{8} \cdot \frac{6}{6} + \frac{3}{8} \cdot \frac{4}{6} + \frac{3}{8} \cdot \frac{2}{6} + \frac{1}{8} \cdot \frac{0}{6} = \frac{24}{48} = \boldsymbol{\frac{1}{2}}.$$

1.B. In this case, too, the probability p_+ that Piglet wins is the sum of probabilities of the same four events, in which Piglet gets, respectively,

$$6 + 0 = 6, \quad 4 + 1 = 5, \quad 2 + 2 = 4, \quad \text{or} \quad 0 + 3 = 3 \text{ points}.$$

and Winnie the Pooh gets a strictly smaller score. This gives the probability

$$p_+ = \frac{1}{8} \cdot \frac{5}{6} + \frac{3}{8} \cdot \frac{4}{6} + \frac{3}{8} \cdot \frac{3}{6} + \frac{1}{8} \cdot \frac{2}{6} = \frac{28}{48} = \boldsymbol{\frac{7}{12}}.$$

2. Similarly to the first two questions, let's calculate Piglet's winning and losing probabilities p_+ and p_- for various $r \in (0, 1)$.

For $0 < r < \frac{1}{3}$, we have

$$p_+ = \frac{1}{8} \cdot \frac{5}{6} + \frac{3}{8} \cdot \frac{4}{6} + \frac{3}{8} \cdot \frac{2}{6} + \frac{1}{8} \cdot \frac{0}{6} = \boldsymbol{\frac{23}{48}} < \boldsymbol{\frac{24}{48}} = \frac{1}{8} \cdot \frac{0}{6} + \frac{3}{8} \cdot \frac{2}{6} + \frac{3}{8} \cdot \frac{4}{6} + \frac{1}{8} \cdot \frac{6}{6} = p_-.$$

For $r = \frac{1}{3}$, we have

$$p_+ = \frac{1}{8} \cdot \frac{5}{6} + \frac{3}{8} \cdot \frac{4}{6} + \frac{3}{8} \cdot \frac{2}{6} + \frac{1}{8} \cdot \frac{0}{6} = \boldsymbol{\frac{23}{48}} = \frac{1}{8} \cdot \frac{0}{6} + \frac{3}{8} \cdot \frac{2}{6} + \frac{3}{8} \cdot \frac{4}{6} + \frac{1}{8} \cdot \frac{5}{6} = p_-.$$

For $\frac{1}{3} < r < 1$, we have

$$p_+ \geq \frac{1}{8} \cdot \frac{5}{6} + \frac{3}{8} \cdot \frac{4}{6} + \frac{3}{8} \cdot \frac{2}{6} + \frac{1}{8} \cdot \frac{1}{6} = \boldsymbol{\frac{24}{48}} > \boldsymbol{\frac{23}{48}} \geq p_-.$$

Problem 2

Pinocchio wants to draw a circle by hand on grid paper lined into squares with side length 1. To this end, he tries to make it go through a number of grid **nodes** (vertices of grid squares). Find the smallest radius of such a circle if:

1) it passes through at least **16** nodes and its center is a grid node;

2) it passes through at least **8** nodes and its center is **not necessarily** a grid node.

Answer: 1) $\sqrt{65}$; **2)** $\sqrt{2{,}5}$.

Solution is based on the Pythagorean Theorem.

1. If a circle centered at a node passes through at least 16 nodes, then the square of its radius must be representable as the sum of two different nonzero squares. The smallest number satisfying this condition is

$$\mathbf{65} = 8^2 + 1^2 = 7^2 + 4^2.$$

The fact that this number is the smallest possible is verified by the direct search through all pairs of positive integers whose squares add up to a number **smaller than 65**. All such sums can be listed by dividing them into five groups of pairs of numbers with the square of the smaller number equal to 1, 4, 9, 16, 25, respectively:

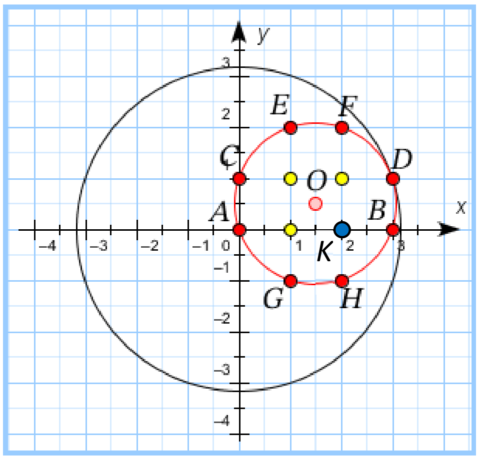
$$5, 10, 17, 26, 37, 50;\ \ 13, 20, 29, 40, 53;\ \ 25, 34, 45, 58;\ \ 41, 52;\ \ 61.$$

We see that any two of these sums are **different**.

2. The circle S on the coordinate plane with center $O(1{,}5; 0{,}5)$ and diameter $\sqrt{1^2+3^2} = \sqrt{10}$ passes through the following **8 grid nodes** (see the figure):

$A(0;0), B(3;0), C(0;1), D(3;1), E(1;2), F(2;2), G(1;-1)$, and $H(2;-1)$.

No circle of a smaller diameter has this property. Indeed, let's assume for definiteness that such a circle S' passes through A. Then it lies inside the circle with center A and radius $AD = \sqrt{10}$, and if it goes through some other node M, then its center O' lies on the perpendicular bisector to segment AM with $AO' < AO = \sqrt{2{,}5}$. Then, by the symmetry of the frame, we can take for M one of the 6 points B, C, E, F, G, or $K(0;2)$, and for the center O', a point in the disk bounded by circle S. It can be seen directly that any such circle passes through less than 8 nodes.



Problem 3

You are given a photograph of a piece of a round bench with a ring-shaped seat made up of identical small planks in the form of an isosceles trapezoid. As any other photograph, it obeys the laws of perspective: straight lines are depicted in it as straight lines, but several straight lines parallel to each other become lines intersecting at a common point on the so-called horizon line (except when they are parallel to the horizon line). Your task is to determine, using these laws and, if necessary, geometric constructions on the sheet with the photograph, the **total number of planks** that constitute the seat.

1. On the photo, construct the **images** of:

A) the line that **bisects** the short sides (bases) of one of the planks;

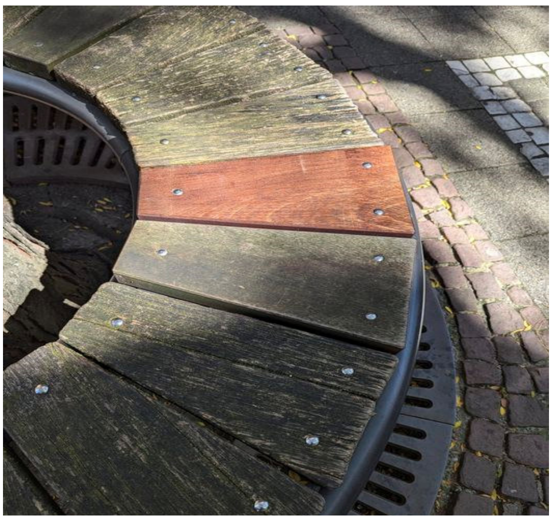
B) a line **parallel** to these two short sides;

C) any two **perpendicular** lines on the bench passing through its center.

(You can show your constructions on a blank sheet.)

2. Find the total number of planks knowing that it is a **multiple of four**.

3. Is it possible to find the total number of planks if **only two** of them are shown in the photo, and we don't know anything about this number in advance?



Answer: 1.A)–C) see Fig. 2; **2.** 24; **3.** Yes.

Solution

Let us recall a well-known theorem: the midpoints of the bases of a trapezoid $XYUV$ lie on the line m joining the intersection points P and O of its diagonals and of its extended sides (Fig. 1). In addition, in the case of an isosceles trapezoid, the line m is the reflection axis of the entire figure.

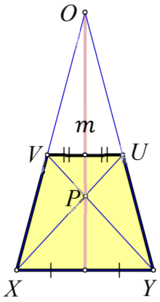


Fig. 1

1. The solution is shown in Fig. 2, where $ABCD$ and $BEFC$ are the images in the photo of two adjacent planks, and segments AB, CD and BE, FC are the images of the bases of the corresponding trapezoids.

A. The answer is the line $\boldsymbol{a}$ in Fig. 2, as immediately follows from the theorem on a trapezoid cited above.

B. The image of any line parallel to the plank's bases, shown by segments AB and CD, passes through the meet K of their extensions. So, the **answer** is any line through K, e.g. $\boldsymbol{OK}$ (in Fig. 2).

C. The image of the center of the bench is the intersection point O of lines AD and BC. Since al the lines perpendicular to the reflection axis of an isosceles trapezoid are parallel to its bases, we get as an **answer**, e.g., the lines $\boldsymbol{a}$ **and** $\boldsymbol{OK}$ (Fig. 2).

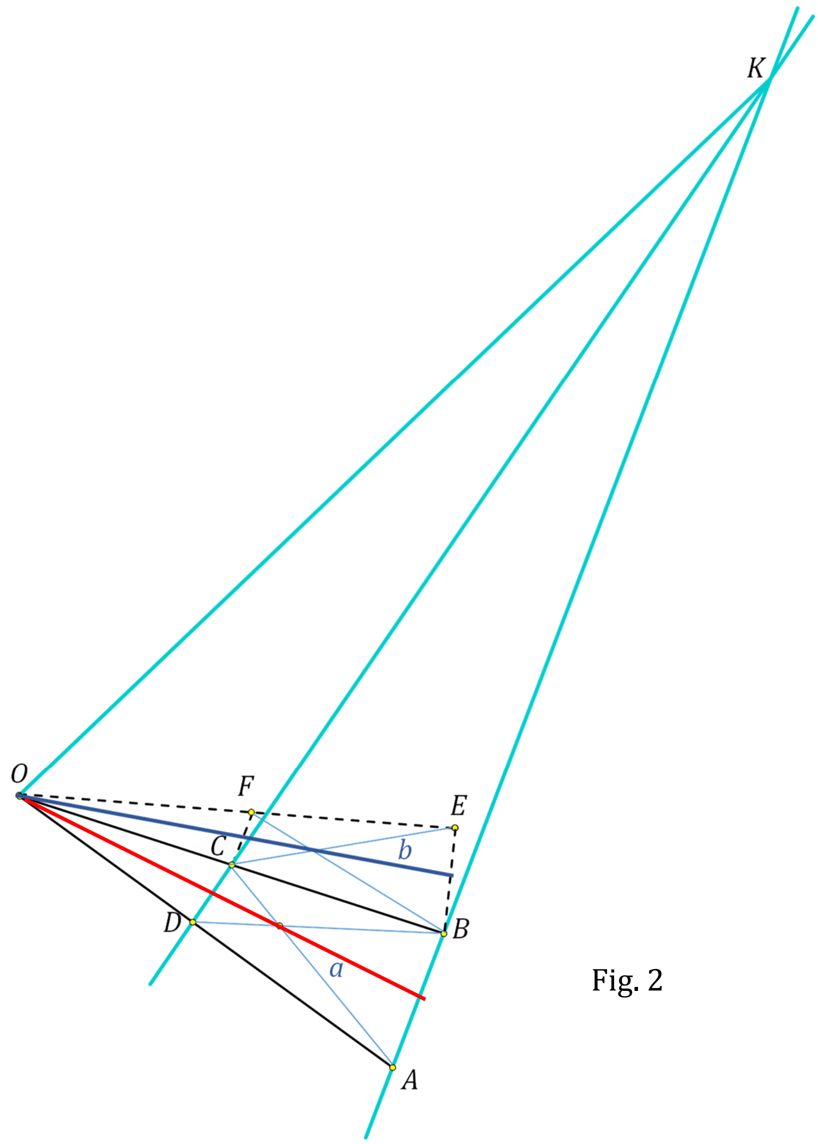


Fig. 2

2. If the number of planks is $\boldsymbol{4n}$, then the angle between the lateral sides of each plank equals $\alpha = 90°/n$. If we draw the lines a and OK on the photo (they make an angle of 90° on the bench), then the angle they form contains 5 entire planks and 2 pieces of planks at the edges of the angle. Therefore $5\alpha < 90° < 7\alpha$, and so $90° = 6\alpha$ and $\boldsymbol{n = 6}$.

3. The construction in Fig. 1 explains how, given a line m and points X and Y symmetric to each other with respect to m and using a **ruler alone**, to construct the point U **symmetric** to an arbitrary point V with respect to m: we construct points O and P at which the axis m meets lines XV and YV, then U is the intersection point of lines XP and YO (in the case where m is parallel to XV or YV, the construction is somewhat more complex, but in this problem this case does not occur). Since all straight lines remain straight lines on the photo, this construction works for the photo as well.

Now suppose that the photo contains the images $ABCD$ and $CDEF$ of two planks (Fig. 2). Construct the reflection axis b on the image of the 2nd plank and, using the points B and E symmetric with respect to b, construct the missing vertices of the 3rd plank **symmetric** to points A and D with respect to line b. Then, in a similar way, we construct the 4th plank, then the 5th one, etc. until the ring closes. The process can be sped up: it will suffice to “travel” only the **semicircle** (to reach the line a); also, one can skip planks (e.g., if we reflect the 1st plank with respect to the reflection axis of the 3rd one; this will give at once give the 5th plank).

Note that we do not even need the entire 2nd plank: it will suffice that the photo contains, in addition to B and C, just one more vertex, say, E. Indeed, since we are given points E and A symmetric with respect to BC, vertex F can be constructed as the point symmetric to D with respect to the same line BC.

4. Hometask. Is it possible to find the number of planks on the bench given the image of **only one** plank?