1. Once Gavrila heated a kettle and pour hot water into in a cylindrical thermocup. He did not cover it so the walls were thermically insulated while the free surface was open to the room. The boy noticed that the water temperature dropped from 90°C to 80°C in 50 seconds. Next time, he poured the same amount of hot water to a mug that has the same size as the thermocup. He firmly closed the cover excluding the heat transfer through the upper surface. In this case, the water temperature dropped from 90°C to 80°C in 30 seconds. How long will the same temperature drop take if the water is placed to the mug and the free surface remains open?

Solution:

We will assume that the power of heat transfer through the walls of the mug and through the free surface (liquid surface) do not depend on whether another mechanism is working at the same time and weakly depends on the temperature of the liquid in the considered temperature range. In this case, the power of heat transfer through the free surface in the thermocupP_1 is equal to the power of heat transfer through the surface of the liquid in the mug, since it is said that they are of the same radius. Let us denote the power of heat transfer through the walls of the cup as P_2. Let us denote the cooling times from the statement in a thermocup and in a mug as τ_1 and τ_2, respectively.

In this case, we have following relations for the amount of heat:

$$cm\Delta T = P_1\tau_1; \quad cm\Delta T = P_2\tau_2;$$
$$cm\Delta T = (P_1 + P_2)\,\tau,$$

where c – water heat capacity, m – mass of water, ΔT – temperature drop.

Thus, from the first two equations one can find the relation between P_1 and P_2, after that from the third and any of the first two one can obtain the relation:

$$\frac{1}{\tau} = \frac{1}{\tau_1} + \frac{1}{\tau_2}$$

$$\tau = \frac{75}{4} = 18{,}75\ c$$

2. A flying robot places small cones over a flat area in such a way that their bases touch each other and centers of the bases lie at concentric circles. In the center of these circles there a big stack of the cones. The robot approaches the stack, takes one cone, flies straight to a proper place at the circle, puts the cone down and flies back for the next cone. The time that is needed to pick the cone up and put it down is negligible relative to the time of flight. The velocity of the flying is constant. The robot keeps putting the cones to a specific circle while there is place for at least one of them at the circle. After that, the robot moves to the next circle that has the radius by a cone base diameter larger.

   a. The robot spent 15 minutes for putting all cones at the circles with the radius of 3 meters. How long will take to put the cones to the circle of the radius of 5 meters?

   b. (senior league only) Maria watched the robot working and noticed that the radius of the circle, where the robot works, depends on time is close to a power-law. What is the exponent of this power?

   Solution:

Arranging the cones along a fixed circle of radius R, the robot makes N trips to the stack and back, each of length 2R. The number of flights is proportional to the length of the circle R, which means that the time spent on a given circle is proportional to R^2. This means that for a circle of radius 5 m it will take

$$T_2 = T_1\left(\frac{R_2}{R_1}\right)^2 = \frac{125}{3} min = 41\, min\, 40\, sec$$

When the robot finishes working on one circle, it moves to another of a close radius, and the change of radius is fixed and equal to the diameter of the cone. Time between transitions from current radius to the next

$$\Delta t \sim R^2,$$

Thus, radius increases according to the following law

$$\frac{\Delta R}{\Delta t} = \frac{A}{R^2},$$

where A is a certain dimension constant m^3/s. The radius of the circle on which the cones are currently placed depends on this constant and on time; there are no other parameters in the problem. So, from considering dimension$R(t) \sim (At)^{\frac{1}{3}}$

Then the required degree is 1/3.

Let us also present the second solution. The radii of the circles are discrete and form an arithmetic progression $R_i - R_{i-1} = const = D_{cone}$. From paragraph awe obtained the relation between the times and radii of any 2 circles, applying it to the very first circle and an arbitrary one, we get the time spent on acircle with number i:

$$T_i = T_1\frac{R_i^2}{R_1^2}$$

In this case, the time that passed before moving to the R_{i+1} radius is equal to the sum of the times

$$T(R_{n+1}) = \sum_{i=1}^{n} T_1\frac{R_i^2}{R_1^2} = \frac{T_1}{R_1^2}\sum_{i=1}^{n} R_i^2 \sim \frac{T_1}{R_1^2}R_n^3$$

The last proportionality is a well-known fact that the sum of the squares of the terms of an arithmetic progression is equal to a polynomial of the third degree of n. That is, we got the same result that time is proportional to the third power of the radius (or vice versa, the radius is proportional to time with an exponent of 1/3), the larger the radius of the circle (or number of the circle), the more accurate this dependence.

$$R_n \sim T^{\frac{1}{3}}$$

It can be considered asymptotic.

3. Vasily bought a photosensor that detects if an object crosses its ray of vision and measures the distance between this object and the sensor. Vasily built a simple launch device and placed in 5 meters from a wall. He threw small marbles towards the wall at the angles between 30° to 60° to the horizon in a vertical plane perpendicular to the wall. Between the launch site and the wall, he placed the sensor at the distance of 0.5 m from

the launch point and directed the ray of vision vertically (see the picture below). Vasiliy wants to use the sensor data to predict the height where the marble hits the wall. He knows that marbles launched with the velocity of 10 m/s and the air drag is negligible.

a. Provide an approximate expression that relates the desired value and the sensor data and evaluate the height of the point hit by a marble that flew at height of 55 cm over the sensor.

b. Estimate the relative error of your result

Solution:

Let us write down the equation of the ball trajectory:

$$y\,(x) = x\,\mathrm{tg}\,\alpha - \frac{gx^2}{2V^2}(1 + \mathrm{tg}^2\,\alpha),$$

where x, y are Cartesian coordinates with the origin at the starting point, the y axis is directed upward, g is the acceleration of gravity, V is the initial velocity of the ball, α is the angle between the initial velocity vector and the horizon. Sensor measuring$y_1 = y(x_1)$ allow us to determine $tg\ \alpha$,x_1 – sensor coordinate. After the tangent of the angle is found, you can calculate the required height as $y_2 = y(x_2)$ where x_2 is the coordinate of the wall. The problem is that to determine the tangent, you need to solve a quadratic equation, the roots of which are then substituted into the quadratic trinomial. The result is a big and hard expression that is difficult to use.

Note that in the considered range of angles the tangent is of the order of unity and the ratio of the second term to the first on the right side of the trajectory equation for $x = x_1$

$$\frac{gx_1}{2V^2} \sim 2.5 \cdot 10^{-2} \ll 1,$$

that is, you can neglect the second term when determining the tangent and get

$$tg\ \alpha \approx \frac{y_1}{x_1}$$

That gives

$$y_2 \approx y_1 \frac{x_2}{x_1} - \frac{gx_2^2}{2V^2}\left(1 + \left(\frac{y_1}{x_1}\right)^2\right)$$

For$y_1 = 0.55$mwe get the answer

$$y_2 \approx 2.75м$$

The accuracy of the calculations is determined by the error in calculating the tangent and rounding the value of g. Let the error in determining the tangent δ:

$$tg\alpha = \frac{y_1}{x_1} + \delta$$

Let's find the absolute error of definition y_2. Since

$$y_2 = y_1 \frac{x_2}{x_1} - \frac{gx_2^2}{2V^2}\left(1 + \left(\frac{y_1}{x_1}\right)^2\right) + \delta\left(x_2 - \frac{gx_2^2}{V^2}\frac{y_1}{x_1}\right) - \delta^2 \frac{gx_2^2}{2V^2}\left(\frac{y_1}{x_1}\right)^2$$

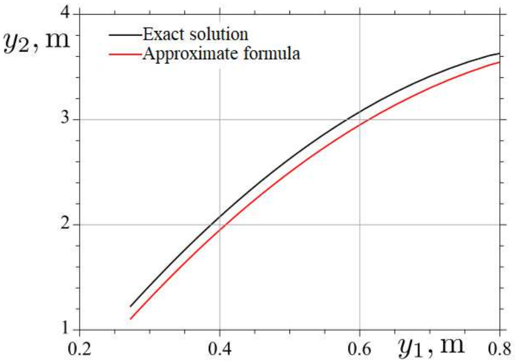
Estimation of absolute error is

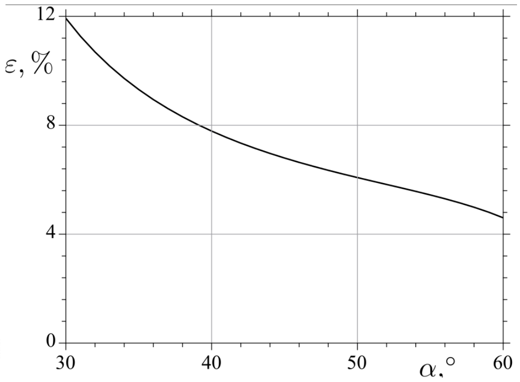
$$\Delta = \left|\delta\left(x_2 - \frac{gx_2^2}{V^2}\frac{y_1}{x_1}\right)\right| \approx \left|tg\ \alpha - \frac{y_1}{x_1}\right|\left|x_2 - \frac{gx_2^2}{V^2}tg\ \alpha\right|$$

And relative error is

$$\varepsilon = \frac{\Delta}{x_2 tg\alpha - \frac{gx_2^2}{2V^2}(1 + tg^2\alpha)}$$

To estimate the calculation error, we take the angle equal to 45°, so that we have $\varepsilon \approx 5\%$ The figures show the required dependence obtained from exact relations and using the proposed estimate, as well as the dependence of the relative error on the angle between the initial speed and the horizon





The resulting error is comparable to that made when rounding the gravitational acceleration to 10 m/s2. In practice, the main error in determining the trajectory of a falling body is caused by air resistance.