Math Around Us
2023/10/31

Problem 1

Flour arrives at a grocery warehouse in big bags. There, it is repackaged into “almost-a-kilogram packs”, i.e. packs whose weight p kg satisfies the condition

$$|p-1| \leq \varepsilon.$$

Find the smallest value of P_0 for which all flour from a bag of any weight P kg (not necessarily an integer), satisfying the estimate

$$P \geq P_0,$$

can be repackaged into a number of almost-a-kilogram packs (maybe not the same weight)

a) for $\varepsilon = 0.007$;

b) for an arbitrary positive $\varepsilon \leq 0.2$.

Answers: **a)** 70,503; **b)** $\left\lceil \frac{1-\varepsilon}{2\varepsilon} \right\rceil (1-\varepsilon)$.

Solution. All possible values of the weight P of any bag that can be repacked into $n \in \mathbb{N}$ almost-a-kilogram packs, fill up the segment

$$K_n = [n(1-\varepsilon), n(1+\varepsilon)].$$

Let’s write down the condition that there is no gap between two neighboring segments K_n and K_{n+1}:

$$n(1+\varepsilon) \geq (n+1)(1-\varepsilon) \;\Leftrightarrow\; n \geq \frac{1-\varepsilon}{2\varepsilon} \;\Leftrightarrow\; n \geq \left\lceil \frac{1-\varepsilon}{2\varepsilon} \right\rceil = n_0,$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. We see that all segments K_n with numbers

$$n \geq n_0 = \left\lceil \frac{1}{2\varepsilon} - \frac{1}{2} \right\rceil \left(\geq \left\lceil \frac{1}{2 \cdot 0{,}2} - \frac{1}{2} \right\rceil = 2 \right)$$

completely fill the half-line $[P_0, +\infty)$, where

b) $P_0 = n_0(1-\varepsilon) = \left\lceil \frac{1-\varepsilon}{2\varepsilon} \right\rceil (1-\varepsilon)$,

while the is a gap between this half-line and the preceding segment K_{n_0-1}, i.e. P_0 is the desired smallest weight of a bag. It follows that

a) for $\varepsilon = 0{,}007$, this value equals

$$P_0 = \left\lceil \frac{1-0{,}007}{2 \cdot 0{,}007} \right\rceil (1-0{,}007) = 71 \cdot 0{,}993 = 70{,}503.$$

Problem 2

A **scissor jack** is a 4-bar linkage with links of equal length (i.e., a rhombus), whose two opposite hinges are connected by a horizontal leadscrew with a handle (see the figure). The other two hinges have platforms attached to them; the lower one rests on the floor or ground, and the upper one supports the load.

The handle is rotated at constant speed, causing the horizontal diagonal of the rhombus to shorten by 1 mm per second, and the vertical diagonal to stretch with the upper platform lifting the load. In the initial position, the jack is flattened so that its top and bottom points are in contact. In the final position, the jack stretches into a vertical column.

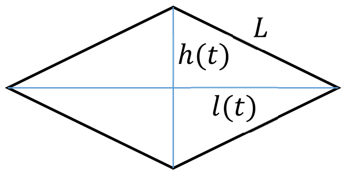
Answer the following questions about the speed at which the top of the jack rises as the handle is rotated:

a) when this speed is maximum: at the start of the motion, at the end of the motion, or somewhere in between;

b) within what limits does this speed vary?

Answer: **a)** at the initial moment; **b)** from $+\infty$ to 0.

Solution. Let $l(t), h(t) \in [0, L]$ be the halves of the horizontal and vertical diagonals of the rhombus, respectively, at the moment $t \in [0, T]$, where $l(0) = L$ is the side length of the rhombus and $l(T) = 0$. Then for the rates of change of these lengths (in mm/s), using the Pythagoras theorem, we get (see the figure)



$$l'(t) = -\frac{1}{2} \Rightarrow h'(t) = \left(\sqrt{L^2 - l(t)^2}\right)' = -\frac{l(t)l'(t)}{\sqrt{L^2 - l(t)^2}} = \frac{l(t)}{2\sqrt{L^2 - l(t)^2}} = \frac{l(t)}{2h(t)}.$$

Since the numerator of the last fraction decreases, the denominator increases, and it is nonnegative, the rate $h'(t)$ decreases as t grows, and so

a) the lifting speed is maximum at the initial moment.

Furthermore, as $t \to 0$, the fraction grows indefinitely (because its denominator tends to 0 while its enumerator is positive and separated from 0), and as $t \to T$, the fraction decreases to 0, i.e.,

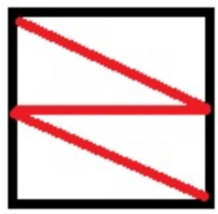
b) the lifting speed $2h'(t)$ of the jack varies from $+\infty$ to 0.

Problem 3

Origami geometric constructions are performed by folding (and then, necessarily, unfolding) a paper square in such a way that the resulting crease:

- either goes through two *previously constructed* points (that is, the intersection points of creases and edges, including the square's corner points);
- or goes through a previously constructed point perpendicular to an edge (after such folding this edge fits on itself);
- or is the perpendicular bisector to a segment with previously constructed endpoints (the paper is folded so that the points coincide).

For example, in the figure at right the first fold (horizontal) yields two congruent rectangles, and two subsequent diagonal folds yield four congruent right triangles.



How many folds do you need for an origami construction on a square of size 1×1 which gives a rectangle of size:

a) $\frac{1}{3} \times 1$;

b) $\frac{1}{5} \times 1$;

c) $\frac{1}{6} \times 1$;

d) $\frac{1}{7} \times 1$?

Try to find constructions with the smallest possible number of folds. Shorter constructions get more points.

Answer: **a)** 4; b**)** 5; c**)** 4; d**)** 5.

Solution. Let's fold the square $ABCD$ in half along the line EF parallel to AB (fold 1), and proceed by folding it along:

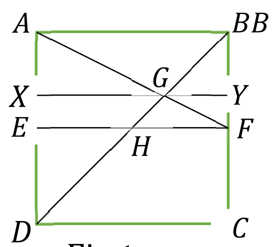


Fig.1

a, c) the diagonal BD (fold 2), the line AF, which meets BD at point G (fold 3), and the line XY passing through G perpendicular to AD (fold 4); then the rectangles $ABYX$ and $EFYX$ (Fig. 1) are the ones required in questions a) and c), respectively, because

$$EX : AX = HF : AB = 1 : 2 \;\; \Rightarrow \;\; AX = \frac{2}{3}AE = \frac{1}{3}AD, \;\; EX = \frac{1}{3}AE = \frac{1}{6}AD;$$

b) the diagonal BD (fold 2), the perpendicular bisector GH to segment AE (fold 3), the line AH, which meets BD at point I (fold 4), and the line XY passong through I perpendicular to AD (fold 5); then the rectangle $ABYX$ (Fig. 2) is the desired one, because

$$GX : AX = JH : AB = 1 : 4 \quad \Rightarrow \quad AX = \frac{4}{5}AG = \frac{1}{5}AD;$$

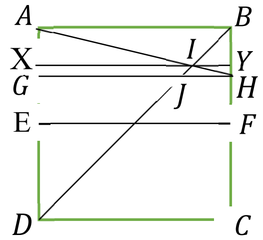


Fig.2

d) the perpendicular bisector GH to segment AE (fold 2), the perpendicular bisector IJ to segment AG (fold 3), the line DJ (Fig. 3), which meets GH at point K (fold 4), and the line XY passing through K perpendicular to AB (fold 5); then the rectangle $BCYX$ is the desired one, because

$$BX : AB = KH : CD = JH : JC = 1 : 7 \quad \Rightarrow \quad BX = \frac{1}{7}AB.$$

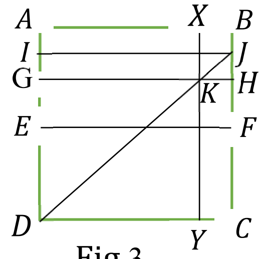


Fig.3

Remark. The construction in Fig. 1 shows how to obtain the point X that cuts off one third of the side AD, given the midpoint E of this side. In the same way we can prove that if $AE = \frac{1}{n}AD$, then the same construction yields $AX = \frac{1}{n+1}AD$. This technique, used repeatedly, enables us to obtain any rectangle $\frac{1}{n} \times 1$ starting from the unit square. This method is universal, but not the shortest.